A High-Order Piecewise Quartic Spline Rule for Hadamard Integral and Its Application of the Cavity Scattering

Meiling Zhao*, Wenjie He and Gendai Gu

Department of Mathematics and Physics, North China Electric Power University, Baoding, China

Abstract. We develop a fourth-order piecewise quartic spline rule for Hadamard integral. The quadrature formula of Hadamard integral is obtained by replacing the integrand function with the piecewise quartic spline interpolation function. We establish corresponding error estimates and analyze the numerical stability. The rule can achieve fourth-order convergence at any point in the interval, even when the singular point coincides with the grid point. Since the derivative information of the integrand is not required, the rule can be easily applied to solve many practical problems. Finally, the quadrature formula is applied to solve the electromagnetic scattering from cavities with different wave numbers, which improves the whole accuracy of the solution. Numerical experiments are presented to show the efficiency and accuracy of the theoretical analysis.

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Key words: Hadamard integral, piecewise quartic spline rule, error estimate, electromagnetic scattering.

1. Introduction

Consider the following integral

$$I(f) = \fint_a^b \frac{f(x)}{(x-s)^2} dx, \quad s \in (a,b), \tag{1.1}$$

where $\fint$ denotes a Hadamard finite-part integral and s is the singular point. The integral operator in (1.1) is called Hadamard integral (or hypersingular integral).

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*Corresponding author. *Email addresses:* `meilingzhao@sina.com` (M. Zhao), `ggd999@126.com` (G. Gu), `wenjiemath@163.com` (W. He)

Hadamard integral widely exists in boundary integral equations and nonlocal boundary conditions related to the fields of electromagnetic scattering, fracture mechanics, continuum mechanics and so on. A large number of boundary integral equations can be obtained by using different hypersingular boundary conditions. In addition, many nonlocal boundary conditions often contain hypersingular integrals. The key to the above problems is the accurate approximation of hypersingular integrals.

Numerous works have been devoted to developing efficient quadrature formulas for Hadamard integrals, such as the Gaussian method [17, 19], the Newton-Cotes method [12, 15, 20, 21], the transformation method [4] and some others [5, 8, 14]. Hadamard [7] first mentioned the concept of the finite-part integral in his study of hyperbolic partial differential equations. Kutt [11] further improved the concept of Hadamard integral and proposed the Gaussian method. Since the integrals are approximated in terms of $f(x)$ at Gaussian points, the Gaussian method is restricted by the mesh selection. In comparison with the Gaussian method, the Newton-Cotes method is easier to implement. Therefore, the Newton-Cotes method is used extensively in practice. Linz [16] first presented the grid-type Newton-Cotes method for evaluating Hadamard integral. However, the grid-type Newton-Cotes method is invalid when the singular point is close to the grid point, even fails completely when the singular point coincides with grid points [16].

Some scholars are committed to developing nodal-type algorithms to approximate the hypersingular integrals in the case that the singular point coincides with grid points. Yu [24] proposed a nodal-type Newton-Cotes method, which allowed the location of the singular point to be unconstrained. The generalized trapezoidal method and the Simpson method were given by adopting the geometric mesh near the singular point in [24], which had $\mathcal{O}(h^2)$ and $\mathcal{O}(h^4)$ convergence rates respectively. Sun [18] also presented a new Newton-Cotes formula which was applicable when the singular point is located at grid points. Afterward, Wu [22] proposed some methods with Toeplitz-type structure for Hadamard finite-part integral operator, which were applied to the electromagnetic scattering from cavities. Then Zhang proved the superconvergence rate of the Newton-Cotes method is higher when the singular point coincides with some priori known points, and the superconvergence rate is $\mathcal{O}(h^{k+1})$ provided that $f(x) \in C^{k+2+\alpha}$ $(0 < \alpha < 1)$ [23, 25].

Following the development of new techniques and new methods, Li [14] presented an extrapolation method to calculate Hadamard integral and obtained the fourth-order approximation when $f(x)$ is smooth enough. Then the generalized middle rectangle rule for the computation of certain hypersingular integrals was discussed [13]. In addition, quadrature rules of splines are important tools for the approximation of integral [2,3,9,27]. The Hermite rule for evaluating Hadamard integrals was also discussed in [27], however, the derivative of the integrand function is required. A cubic spline rule for the evaluation of Hadamard finite-part integral was presented in [6]. The convergence order of the cubic spline rule is $\mathcal{O}(h^3)$. When the singular point coincides with a prior known point, the convergence rate is one order higher than what is globally possible. Nevertheless, this rule is inefficiency in the case that the singular points