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Painlevé Property and Integrability of Polynomial Dynamical Systems

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Abstract: The main purpose of this paper is to investigate the connection between the Painlevé property and the integrability of polynomial dynamical systems. We show that if a polynomial dynamical system has Painlevé property, then it admits certain class of first integrals. We also present some relationships between the Painlevé property and the structure of the differential Galois group of the corresponding variational equations along some complex integral curve.

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1 Introduction

The connection between the integrability of dynamical systems and the singularity structure of their solutions was first discovered since the pioneer work of Kowalevskaya[1] in studying the problem for the motion under gravity of a rigid body about a fixed point. The remarkable insight of Kowalevskaya is that the concerned problem can be solved explicitly whenever the parameters are taken such that the dependent variables are meromorphic with respect to the time t in the complex plane. It is well known that the solvable cases are the four integrable cases of the Heavy top problem (see [2]): isotropic, Euler, Lagrange and Kowalevskaya. The idea of Kowalevskaya was trying to detect the regular property of dynamical systems by studying the singular structure of their solutions. So far, one of the famous singular analysis results is the Painlevé analysis, which detects the Painlevé property of a given system established by Painlevé and some other mathematicians following his work. Generally

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speaking, a dynamical system is integrable if it has sufficient number of first integrals such that its general solution can be solved by quadratures, while a given system has Painlevé property if its general solution has no movable critical singular points on the complex plane. Since 80th decade of the last century, there have been numerous references and reports for the successful developments and applications of Painlevé analysis, such as [3–10] and so on. Among of these results, an interesting phenomenon is that many concerned systems passing the Painlevé test always turn to be integrable, which implies the deep relationships between integrability and Painlevé property, and a natural question should be asked that, what is the exact connection between the integrability and the Painlevé property for general dynamical systems? One of the famous results is due to Yoshida[11–12] who studied the connection between the integrability of dynamical systems and corresponding Kowalevskaya exponents with respect to a balance. Following Yoshida's work, many results have been obtained on the connection between the integrability, partial integrability, non-integrability of dynamical systems and corresponding Kowalevskaya exponents, see [13–16] for example. However, nearly all these results are the necessary conditions for the integrability, partial integrability and non-integrability of dynamical systems, i.e., the idea of these results are from integrability to the singularity property.

About several years after the Kowalevskaya's work, Liapounov[2] proved that the only cases of the family of Heavy top problem with a general solution uniform over the whole complex time plane are the four integrable cases (see [2]). In this way Liapounov improved the Kowalevskaya's result, he considered for the first time the Poincaré's variational equation of the concerned system, with respect to the initial conditions, in the complex time along a suitable particular solution. In 1982, Ziglin[17] got a non-integrability result for complex analytical Hamiltonian systems by using the constrains imposed by the existence of some first integrals on the monodromy group of the normal variational equation along some complex integral curve. This idea was later developed by Morales-Ruiz *et al.*[18–19] and Baider *et al.*[20] in the end of the last century. By using the differential Galois theory and investigating the relation of the integrability and the structure of the identity component of the differential Galois group of the variational equation along some complex integral curve, they obtained a stronger integrability condition for Hamiltonian systems, the corresponding method and results are usually called Morales-Ramis theory. So far, many developments and applications of differential Galois theory to the integrability have been obtain in numerous references and reports, such as [18–19, 21–25] and so on.

In the present paper, we firstly investigate the connection between the Painlevé property and the integrability of polynomial dynamical systems, and beyond the known results, we get the existence of certain class of integrals in the assumption of Painlevé property, we also investigate the connection between the Painlevé property of polynomial dynamical systems and the structure of the differential Galois group of the corresponding variational equations along some complex integral curve.

The structure of this paper is as follows. In Section 2, we firstly recall the preliminary procedure and some results of Painlevé analysis, and the relationships between the Painlevé