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The Maximum Trigonometric Degrees of Quadrature Formulae with Prescribed Nodes

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Abstract: The purpose of this paper is to study the maximum trigonometric degree of the quadrature formula associated with m prescribed nodes and n unknown additional nodes in the interval $(-\pi, \pi]$. We show that for a fixed n, the quadrature formulae with m and $m+1$ prescribed nodes share the same maximum degree if m is odd. We also give necessary and sufficient conditions for all the additional nodes to be real, pairwise distinct and in the interval $(-\pi, \pi]$ for even m, which can be obtained constructively. Some numerical examples are given by choosing the prescribed nodes to be the zeros of Chebyshev polynomials of the second kind or randomly for $m \geq 3$.

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1 Introduction

Let $\omega(\theta)$ be a non-negative weight function on $(-\pi, \pi]$, vanishing there only on a set of a measure zero. For the integration of the form

$$I[f] = \int_{-\pi}^{\pi} \omega(\theta) f(\theta) \mathrm{d}\theta, \tag{1.1}$$

a quadrature formula

$$Q_n[f] = \sum_{j=1}^{n} \alpha_j f(\theta_j), \qquad \theta_j \neq \theta_k, \ j \neq k \tag{1.2}$$

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is said to have trigonometric degree d if $Q_n[f] = I[f]$ for any $f \in \mathcal{T}_d^0$ and for at least one $g \in \mathcal{T}_{d+1}^0$ such that $Q_n[g] \neq I[g]$, where

$$\mathcal{T}_d^0 = \text{span}\{1, \cos k\theta, \sin k\theta \mid k = 1, \cdots, d\}.$$

Typically, θ_j and α_j are called nodes and weights of $Q_n[f]$, respectively.

Recently, the following problem has been arisen:

Problem 1.1 Given m distinct points $\{x_j\}_{j=1}^m \subset (-\pi, \pi]$ and $n \in \mathbf{N}$, find n distinct points $\{y_k\}_{k=1}^n \subset (-\pi, \pi]$ and positive numbers $\{A_j\}_{j=1}^m$, $\{B_k\}_{k=1}^n$ such that the quadrature formula

$$Q_{m,n}[f] = \sum_{j=1}^{m} A_j f(x_j) + \sum_{k=1}^{n} B_k f(y_k) \tag{1.3}$$

has maximum trigonometric degree.

Problem 1.1 has been dealt with when $m = 1$ and $m = 2$ by Jagels and Reichel[1] and Butheel *et al.*[2], using the technique of complex analysis. These formulae are called the Szegö-Radau ($m = 1$) and Szegö-Lobatto ($m = 2$) quadrature formulae, respectively.

In this paper, we mainly study the maximum trigonometric degree of (1.3) for any positive integer m. The maximum degree is denoted by $d_{\max}(m, n)$ and given in Theorem 2.1. Some simple computation shows that for a fixed n, the quadrature formulae with m and $m+1$ prescribed nodes share the same degree when m is odd. Furthermore, for even m, with the theory of Complex K-Moment Problem (CMP) (see [3–9] and references therein), we obtain necessary and sufficient conditions of the existence of y_k having the property that they are real, pairwise distinct and in the interval $(-\pi, \pi]$. If such conditions are satisfied, the nodes y_k and all the weights can be computed constructively. To facilitate the description, the nodes satisfying the above property are called simple nodes and the quadrature formulae of the form (1.3) with simple nodes y_k and positive weights A_j, B_k are called desired quadrature formulae.

The paper is organized as follows. The maximum trigonometric degree of (1.3) is studied in Section 2 for any positive integer. The necessary and sufficient conditions of the existence of simple additional nodes are discussed in Section 3. Some examples are shown in Section 4. A brief conclusion is given in Section 5.

2 Maximum Trigonometric Degrees

Given N distinct points $\{\theta_j\}_{j=1}^N \subset (-\pi, \pi]$, let $\Theta = \{\theta_j\}_{j=1}^N \subset (-\pi, \pi]$, $N = 2(n_0 + \gamma)$, $\gamma \in \left\{0, \dfrac{1}{2}\right\}$. Denote $\mathcal{T}^0 = \bigcup\limits_{d \in \mathbf{N}} \mathcal{T}_d^0$. We define

$$\mathfrak{I}_\Theta = \{f(\theta) \in \mathcal{T}^0 \mid f(\theta_j) = 0,\ j = 1, 2, \cdots, N\},$$

and

$$S_{\mathfrak{I}_\Theta} = \{f(\theta) \in \mathcal{T}^0 \mid f(\theta_j) \neq 0,\ \text{at least for one } j \in \{1, 2, \cdots, N\}\}.$$