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Co-splitting of Simple Lie Algebras of Type A, D, E

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Abstract: In this paper, through a meticulous description of finite root system, a concrete comultiplication with an explicit action on the basis elements of finite dimensional simple Lie algebras of type A, D, E is constructed. Then any finite dimensional simple Lie algebra of type A, D, E is endowed with a new generalized Lie coalgebra splitting. This construction verifies the known existence of a co-split Lie structure on any finite dimensional complex simple Lie algebra.

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1 Introduction

During the past decade, a great number of papers study Lie bialgebras. It is well-known that a Lie bialgebra is a vector space endowed simultaneously with a Lie algebra structure and a Lie coalgebra structure, together with a certain compatibility condition, which was suggested by a study of Hamiltonian mechanics and Poisson Lie groups (see [1]).

Recently, Xia and Hu[2] introduced a new concept "co-split Lie algebra" which is a new [Lie algebra]-[Lie coalgebra] structure, and proved that any finite dimensional complex simple Lie algebra L can be endowed with a co-split Lie structure, i.e., a co-splitting Lie coalgebra structure such that the composition $[\ \cdot\ ,\ \cdot\ ] \circ \delta$ of the two structure maps $\delta : L \to L \otimes_{\mathbf{C}} L$ and $[\ \cdot\ ,\ \cdot\ ] : L \otimes_{\mathbf{C}} L \to L$ coincides with the identity. Using the concept "co-split Lie algebra", the Lie algebra structure on the dual space of a semi-simple Lie algebra can be easily studied from another point of view. Moreover, Farnsteiner[3] elicited the conceptual sources of [2], starting from the observation that the coalgebra maps defined in [2] are in

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fact homomorphisms of L-modules, and for Lie algebras affording non-degenerate symmetric associative forms, such coalgebra maps naturally arise by dualizing the Lie multiplication, also several equivalent characterizations of co-splitting of a Lie algebra are given. For the co-split Lie algebra L of type A_l, Xia and Hu[2] have shown an explicit action of the coalgebra map δ on the basis elements of L. For the co-split Lie algebra L of another type, δ is obtained via embedding $L \hookrightarrow sl_n(\mathbf{C})$ and the action of δ on the basis elements of L is not explicitly shown, and may be complicated. In this paper, a co-split Lie algebra structure is given, which generalizes the construction in Theorem 4.2 in [2], on any complex simple Lie algebra L of type A_l $(l \geq 1)$, D_l $(l \geq 4)$ or E_k $(k = 6, 7, 8)$, and the coalgebra map δ has an explicit action on the basis elements of L. See Theorem 4.1 in this paper for details.

2 Basic Definitions and Notations

A Lie algebra is a pair $(L, [\,\cdot\,,\,\cdot\,])$, where L is a linear space and $[\,\cdot\,,\,\cdot\,] : L \otimes_{\mathbf{C}} L \to L$ is a bilinear map (in fact, it is a linear map from $L \otimes_{\mathbf{C}} L$ to L) satisfying

(L1) $[a, b] + [b, a] = 0$;

(L2) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

For any spaces U, V, W, define linear maps $\tau : U \otimes_{\mathbf{C}} V \to V \otimes_{\mathbf{C}} U$ by $\tau(u \otimes v) = v \otimes u$, and $\xi : U \otimes_{\mathbf{C}} V \otimes_{\mathbf{C}} W \to V \otimes_{\mathbf{C}} W \otimes_{\mathbf{C}} U$ by $\xi(u \otimes v \otimes w) = v \otimes w \otimes u$. A Lie coalgebra is a pair (L, δ), where L is a linear space and $\delta : L \to L \otimes_{\mathbf{C}} L$ is a linear map satisfying

(Lc1) $(1 + \tau) \circ \delta = 0$;

(Lc2) $(1 + \xi + \xi^2) \circ (1 \otimes \delta) \circ \delta = 0$.

A Lie bialgebra is a triple $(L, [\,\cdot\,,\,\cdot\,], \delta)$ such that

(Lb1) $(L, [\,\cdot\,,\,\cdot\,])$ is a Lie algebra;

(Lb2) (L, δ) is a Lie coalgebra;

(Lb3) For any $x, y \in L$, $\delta([x, y]) = x \cdot \delta(y) - y \cdot \delta(x)$.

The compatibility condition (Lb3) shows that δ is a derivation map. In this case, $[\,\cdot\,,\,\cdot\,] \circ \delta$ is a derivation of L. Xia and Hu[2] replaced the above (Lb3) with the condition $[\,\cdot\,,\,\cdot\,] \circ \delta = \mathrm{id}_L$ and give the following new concept "co-split Lie algebra".

Definition 2.1 *Suppose that $(L, [\,\cdot\,,\,\cdot\,])$ is a Lie algebra and (L, δ) is a Lie coalgebra. A triple $(L, [\,\cdot\,,\,\cdot\,], \delta)$ is called a co-split Lie algebra if $[\,\cdot\,,\,\cdot\,] \circ \delta = \mathrm{id}_L$.*

3 Several Properties of Simple Lie Algebras of Type A, D, E

Let Q be the root lattice of type A_l, D_l, or E_l, and let $(\,\cdot \mid \cdot\,)$ be the bilinear symmetric form on Q such that the root system $\Phi = \{\alpha \in Q \mid (\alpha \mid \alpha) = 2\}$. Let $\varepsilon : Q \times Q \to \{\pm 1\}$ be an asymmetry function satisfying the bimultiplicativity condition

$$\varepsilon(\alpha + \alpha', \beta) = \varepsilon(\alpha, \beta)\varepsilon(\alpha', \beta), \quad \varepsilon(\alpha, \beta + \beta') = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \beta'), \qquad \alpha, \alpha', \beta, \beta' \in Q,$$