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Pseudopolarity of Generalized Matrix Rings over a Local Ring

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Abstract: Pseudopolar rings are closely related to strongly π-regular rings, uniquely strongly clean rings and semiregular rings. In this paper, we investigate pseudopolarity of generalized matrix rings $K_s(R)$ over a local ring R. We determine the conditions under which elements of $K_s(R)$ are pseudopolar. Assume that R is a local ring. It is shown that $\boldsymbol{A} \in K_s(R)$ is pseudopolar if and only if $\boldsymbol{A}$ is invertible or $\boldsymbol{A}^2 \in J(K_s(R))$ or $\boldsymbol{A}$ is similar to a diagonal matrix $\begin{bmatrix} u & 0 \\ 0 & j \end{bmatrix}$, where $l_u - r_j$ and $l_j - r_u$ are injective and $u \in U(R)$ and $j \in J(R)$. Furthermore, several equivalent conditions for $K_s(R)$ over a local ring R to be pseudopolar are obtained.

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1 Introduction

Throughout this paper all rings are associative with unity. We adopt the following notations from Koliha and Patricio[1]. For an element $a \in R$, the commutant and double commutant of a in R are defined by

$$\mathrm{comm}_R(a) = \{x \in R : ax = xa\}$$

and

$$\mathrm{comm}^2_R(a) = \{x \in R : xy = yx \ \text{ for all } y \in \mathrm{comm}_R(a)\},$$

respectively, if there is no ambiguity, we simply use $\mathrm{comm}(a)$ and $\mathrm{comm}^2(a)$ for short. Let

$$R^{\mathrm{qnil}} = \{a \in R : 1 + ax \in U(R) \text{ for every } x \in \mathrm{comm}(a)\}.$$

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If $a \in R^{\text{qnil}}$ then a is said to be quasinilpotent (see [2]). An element a is quasipolar (see [1]) if there exists a $p \in R$ with $p = p^2$ such that

$$p \in \text{comm}_R^2(a), \qquad a + p \in U(R), \qquad ap \in R^{\text{qnil}}. \tag{$*$}$$

Any idempotent p satisfying the above conditions is called a spectral idempotent of a, which is uniquely determined by the element a if it exists. This term is borrowed from spectral theory in Banach algebras. A ring is said to be quasipolar (see [3]) if every element in R is quasipolar. It was proved in [3] that all local rings and strongly π-regular rings are quasipolar and quasipolar rings are strongly clean. If the condition $ap \in R^{\text{qnil}}$ in $(*)$ is replaced by $a^k p \in J(R)$ for $k \geq 1$, then the element $a \in R$ is called pseudopolar (see [4]), and in this case, the idempotent p is called a strongly spectral idempotent of a and denoted by a^{Π}. R is called pseudopolar if all elements of R are pseudopolar. It was shown in [5] that both uniquely strongly clean rings and strongly π-regular rings are pseudopolar, and that pseudopolar rings are quasipolar. It was also proved in [5] that for abelian rings, pseudopolar rings coincide with semiregular rings.

Let R be a ring, and $s \in R$ be central. Following Krylov[6], we use $K_s(R)$ to denote the set $\{[a_{ij}] \in M_2(R) \mid a_{ij} \in R\}$ with operations as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa'+sbc' & ab'+bd' \\ ca'+dc' & scb'+dd' \end{bmatrix}.$$

The element s is called the multiplier of $K_s(R)$. The set $K_s(R)$ becomes a ring with these operations and can be viewed as a special kind of Morita context. A Morita context (A, B, M, N, ψ, ϕ) consists of two rings A and B, two bimodules ${}_AM_B$, ${}_BN_A$ and a pair of bimodule homomorphisms

$$\psi : M \otimes_B N \to A, \qquad \phi : N \otimes_A M \to B,$$

which satisfy the following associativity:

$$\psi(m \otimes n)m' = m\phi(n \otimes m'), \quad \phi(n \otimes m)n' = n\psi(m \otimes n'), \qquad n, n' \in N,\ m, m' \in M.$$

These conditions imply that the set T of generalized matrices $\begin{bmatrix} a & m \\ n & b \end{bmatrix}$; $a \in A$, $b \in B$, $m \in M$, $n \in N$ forms a ring, called the ring of the Morita context. A Morita context $\begin{bmatrix} A & M \\ N & B \end{bmatrix}$ with $A = B = M = N = R$ is called a generalized matrix ring over R. It was observed by Krylov and Tuganbaev[7] that the generalized matrix rings over R are precisely these rings $K_s(R)$ with $s \in C(R)$. $K_1(R)$ is just the matrix ring $M_2(R)$, but $K_s(R)$ can be significantly different from $M_2(R)$. In fact, for a local ring R and $s \in C(R)$, $K_s(R) \cong K_1(R)$ if and only if $s \in U(R)$ (see Lemma 3 and Corollary 2 in [6], and Corollary 4.10 of [8]).

Some properties of the ring $K_s(R)$ were studied comprehensively in [7]. And in [9–10] the strong cleanness of the generalized matrix ring $K_s(R)$ over a local ring was studied. The quasipolarity of the generalized matrix ring $K_s(R)$ over a commutative local ring was discussed in [11].