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Note on Finding an Optimal Deflation for Quadratic Matrix Polynomials

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Abstract. This paper is concerned with the way to find an optimal deflation for the eigenvalue problem associated with quadratic matrix polynomials. This work is a response of the work by Tisseur et al., *Linear Algebra Appl., 435:464-479, 2011*, and solves one of open problems raised by them. We build an equivalent unconstrained optimization problem on eigenvalues of a hyperbolic quadratic matrix polynomial of order 2, and develop a technique that transforms the quadratic matrix polynomial to an equivalent one that is easy to solve. Numerical tests are given to illustrate several properties of the problem.

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1 Introduction

Given a quadratic matrix polynomial

$$Q(\lambda) = \lambda^2 M + \lambda C + K,$$

where $M,C,K \in \mathbb{R}^{n\times n}$ with M nonsingular. Its associated quadratic eigenvalue problem is

$$Q(\lambda)x = 0, \quad y^{\mathrm{H}} Q(\lambda) = 0,$$

where λ is an eigenvalue and x,y are its corresponding (right) eigenvector and left eigenvector respectively. An eigenvalue is of positive type, if $y^{\mathrm{H}} Q'(\lambda)x = y^{\mathrm{H}}(2\lambda M + C)x > 0$; An eigenvalue is of negative type, if $y^{\mathrm{H}} Q'(\lambda)x = y^{\mathrm{H}}(2\lambda M + C)x < 0$.

Suppose that $\lambda(Q)$, the spectra of $Q(\lambda)$, is $\{\lambda_1, \cdots, \lambda_{2n}\}$. Deflating two distinct eigenvalues λ_1, λ_2 is to construct a new quadratic matrix polynomial

$$\widetilde{Q}(\lambda) = \begin{bmatrix} Q_d(\lambda) & \\ & q(\lambda) \end{bmatrix} = \lambda^2 \begin{bmatrix} M_d & \\ & m \end{bmatrix} + \lambda \begin{bmatrix} C_d & \\ & c \end{bmatrix} + \begin{bmatrix} K_d & \\ & k \end{bmatrix}$$

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such that $\lambda(q)=\{\lambda_1,\lambda_2\}$, $\lambda(Q_d)=\{\lambda_3,\cdots,\lambda_{2n}\}$. Usually, in the two eigenvalues, one is of positive type, and the other is of negative type. If M,C,K are symmetric, and $\lambda_1\in\lambda(Q)$ but λ_1 is nonreal, then $\overline{\lambda_1}\in\lambda(Q)$, and in this case, it is usually required to deflate this conjugate pair together.

The deflation technique is very useful and popular in computing the eigenvalues of a matrix, so that it is hoped to be used for computing the eigenvalues of quadratic matrix polynomials. However, as far as we know, not many works discussed on this topic. Meini [5] discussed a deflation method coupled in her so-called "shift-and-deflate" technique. Tisseur et al. [6] presented a general way to deflate two distinct eigenvalues of quadratic matrix polynomials if the corresponding eigenvectors are given.

Here we briefly describe the idea of the method introduced by Tisseur et al. First they developed a method to deflate a quadratic polynomial for two given eigenvalues whose eigenvectors are parallel. Then they invented a way to transform a quadratic polynomial for two given eigenvalues whose eigenvectors are nonparallel into a new one that has two eigenvalues whose eigenvectors are parallel, i.e., transform this case into the solved case.

The new quadratic matrix polynomial produced by the deflation may have a significantly large condition number compared to the original quadratic matrix polynomial. For the special case that M,C,K are symmetric, Tisseur et al. gave an optimal choice to minimize the condition number for the parallel case, but for the nonparallel case, in [6, Section 3], they reported:

> Identifying which solution minimizes the condition number $\kappa_2(T)=\|T\|_2\|\|T^{-1}\|_2$ remains an open problem.

Here T is a related transformation matrix, of which the detailed form will be given below.

The aim of this paper is to solve this problem. First this problem is formulated and simplified in Section 2, which induces a constrained optimization problem on the eigenvalues of a hyperbolic quadratic matrix polynomial of order 2. Next we parameterize (or equivalently nondimensionalize) it and obtain an unconstrained optimization problem in Section 3. Then we calculate the gradient and the Hessian matrix of the objective function in Section 4. Then we make several numerical tests to show the properties of this problem and suggest a technique that transforms it to an equivalent problem whose objective function is easy to solve, as is shown in Section 5. Finally some concluding remarks is given in Section 6.

Notation. Throughout this paper, I_n (or simply I if its dimension is clear from the context) is the $n\times n$ identity matrix. For any scalar, vector, or matrix X, $\Re X$ and $\Im X$ are its real part and imaginary part respectively; while $\|X\|_2$ and $\|X\|_\infty$ are its spectral norm and sum-of-row norm. For any matrix X, $\lambda(X)$ represents its spectra, and $\lambda_*(X)$ represents the set consisting of all its nonzero eigenvalues. For any real symmetric matrix X, $X\succ 0$ ($X\succeq 0$) means that X is positive (semi-)definite, and $X\prec 0$ ($X\preceq 0$) if $-X\succ 0$ ($-X\succeq 0$).