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Multiplicity and Stability of Equilibrium States of Three-Dimensional Nonlinear System*

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Abstract The multiplicity and stability of the equilibrium states of a three-dimensional differential system with initial conditions and three cross terms are studied in this paper. The existence and multiplicity of equilibrium states are given under the different qualifications of parameters. Besides, the local stability of the equilibrium state is shown by analyzing the eigenfunction of the Jacobi matrix. The global stability of the equilibrium state is obtained by constructing the Lyapunov function. Furthermore, the numerical simulation intuitively reflected the relationship of variables and verified the correctness of theoretical analysis.

Keywords Equilibrium states, multiplicity, local stability, global stability, numerical simulation.

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1. Introduction

Kinematics, dynamics problem involving force in physics can be analyzed by ordinary differential equation. In addition, ordinary differential equations have been widely used in the fields of chemistry, biology, economics, and demographics. People realized the importance of differential equations [7] and the powerful role of mathematical deduction until the Neptune is discovered.

The development of differential equations has gone through the classical stage, the stage of well-posedness theory, the stage of analytical theory and the stage of qualitative theory. As we all know, Poincaré put forward and studied the qualitative theory of differential equations in the 19th century. And the study of stability theory is initiated by Lyapunov. For centuries, mathematicians have never stopped studying the existence, stability, and well-posedness of solutions of differential equations([8, 9, 11, 12, 14–16]). Fortunately, the content of this aspect is constantly improved. The solution that makes the derivative of the differential system equal to zero is called equilibrium state. Equilibrium state is not an abstract mathematical concept. The examples are more common in life. The highest point and the lowest point of the single pendulum motion in the mechanical system, the coexistence of

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the two groups in the ecosystem which quantity remains unchanged, the unchanged supply and demand in the economic system, the unchanged concentration of the substance in the chemical reaction and so on. The equilibrium state of the differential system is one of the most important states of the system. Moreover, the stability of the system can be directly reflect by equilibrium state. The study of the equilibrium state of the infectious disease model can better guide us to prevent, treat, and control the infection of the disease; the study of the equilibrium state of the predator-prey model contributes to the sustainable development of the environment; the study of the equilibrium state of the fishing model can obtain greater economic benefits under the premise of ecological balance. So the study of equilibrium stability has important practical significance which attracts many mathematics scholars. About the theoretical research on equilibrium state, one can refer to [1–6, 10, 13].

The stability of equilibrium states of linear systems can be shown by solving the eigenvalues of Jacobi matrices. However, the stability of the equilibrium state of nonlinear systems is complicated with the appearance of cross terms, high order terms, and so on. Inspired by the reference [5, 8, 10, 12, 16], we studied the following problem

$$\begin{cases} \frac{dx}{dt} = uN + \alpha z - \beta xy - \gamma xz - (u+\xi)x, \\ \frac{dy}{dt} = \beta xy - \epsilon yz - (\delta + u)y, \\ \frac{dz}{dt} = \epsilon yz + \delta y + \gamma xz + \xi x - (\alpha + u)z, \\ x + y + z = N, \\ x(0) = x_0 > 0,\ y(0) = y_0 > 0, z(0) = z_0 > 0, \end{cases} \tag{1.1}$$

where u, α, β, γ, ξ, ε are positive parameters. Particularly, (1.1) can be reduced to infectious disease model with vaccination when $\gamma = \epsilon = 0$.

We shall apply eigenvalue theory and Lyapunov function [5, 10] to obtain the local stability and global stability of three-dimensional differential dynamic system with initial value conditions and nonlinear terms. With the xy, yz, xz taken into consideration, difficulties such as how to deal with the complex eigenfunction with multiple parameters and how to structure the Lyapunov function have to be overcome.

This paper is organized as follows. In Section 2, some notations are given which are critical to main results. In Section 3, existence and multiplicity of equilibrium state are shown. In Section 4, the local stability of equilibrium state is shown by analyzing the eigenfunction of Jacobi matrix. In Section 5, the global stability of equilibrium is obtained by the Lyapunov function. In Section 6, the numerical simulation is presented to illustrate the correctness and realizability of our theoretical results.

2. Some notations and Lemmas

To illustrate the main results, we give the following notations and lemmas.
$\Delta = (\alpha + \gamma N + u + \xi)^2 - 4\gamma(uN + \alpha N)$,
$\Delta_1 = b^2 - 4ac$, $a = \beta(\beta + \epsilon - \gamma)$,
$b = -u\epsilon + \beta\alpha - \epsilon\xi + \gamma(\delta + u) - \beta\delta - u\beta - N\epsilon\beta$,
$c = uN\epsilon - (\delta + u)(\alpha + u) + u\delta + u^2$,