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Ergodic Behaviour of Nonconventional Ergodic Averages for Commuting Transformations*

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Abstract Based on T. Tao's celebrated result on the norm convergence of multiple ergodic averages for commuting transformations, we find that there is a subsequence which converges almost everywhere. Meanwhile, we obtain the ergodic behaviour of diagonal measures, which indicates the time average equals the space average. According to the classification of transformations, we also give several different results. Additionally, on the torus $\mathbb{T}^d$ with special rotation, we prove the pointwise convergence in $\mathbb{T}^d$, and get a result for ergodic behaviour.

Keywords Commuting transformation, convergence almost everywhere, ergodic behaviour, time average, space average.

MSC(2010) 37A20, 28A35, 47E05.

1. Introduction

In 2008, T. Tao proved a convergence result for several commuting transformations:

Theorem 1.1. [14] Let $d \geq 1$ be an integer. Assume that $T_1, T_2, \ldots, T_d : X \to X$ are commuting invertible measure-preserving transformations of a measure space $(X, \mathcal{B}, \mu)$. Then, for any $f_1, f_2, \ldots, f_d \in L^\infty(X, \mathcal{B}, \mu)$, the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T_1^n x) \ldots f_d(T_d^n x) \tag{1.1}$$

are convergent in $L^2(X, \mathcal{B}, \mu)$.

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Soon after, H. Towsner [15], B. Host [9] and T. Austin [2] gave proofs of Theorem 1.1 from different viewpoints. T. Tao's approach was combinatorial and finitary, inspired by the hypergraph regularity and removal lemmas. H. Towsner used non-standard analysis, whereas T. Austin and B. Host all exploited ergodic methods, building an extension of the original system with good properties.

There is a rich history towards Theorem 1.1. For $d = 1$, it reduces to the classical mean ergodic theorem. When $T_1 = T, T_2 = T^2, \dots, T_d = T^d$, Furstenberg studied such averages originally in his proof of Szemerédi's theorem [7], where T is weakly mixing or T is general but $d = 2$. For higher d, various special cases have been shown by Conze and Lesigne [4, 5], Furstenberg and Weiss [8], Host and Kra [10], and Ziegler [19]. Finally, it was totally proved by Host and Kra [11] for arbitrary d, and independently by Ziegler [20].

When $T_1, T_2, \dots, T_d$ are commuting measure-preserving transformations with some hypothesis on the transformations, Zhang [18] gave a proof for $d = 3$ and Frantzikinakis and Kra [6] for general d. Without those assumptions, the L^2-convergence of the averages (1.1) was established by Tao. As we have mentioned above, it possesses four different proofs. When $T_1, T_2, \dots, T_d$ belongs to nilpotent group, it was proved by Miguel N. Walsh [16].

Although most people believe the existence of the averages (1.1) almost everywhere, the cases in which one knows the answer are scarce. In this paper, With the fact that the averages (1.1) have a subsequence which converges almost everywhere, the ergodic behaviour of diagonal measure is proved. Furthermore, on the torus $\mathbb{T}^d$ with special rotation, say, $R_{\alpha_1,\dots,\alpha_d} : \mathbb{T}^d \to \mathbb{T}^d$, where $1, \alpha_1, \dots, \alpha_d$ are rationally independent, the convergence of the averages (1.1) for every point in $\mathbb{T}^d$ is obtained, and a result for ergodic behaviour is presented.

Before launching into the main result, we first remind the reader some elements of the measure theory and the ergodic theory in Section 2. With sufficient preparation, we give a proof of the ergodic behaviour of Theorem 1.1, and give a classification of $T_1, T_2, \dots, T_d$, in case 1: all the T_i are pairwise different, i.e., $T_i \neq T_j,\ i \neq j$, and in case 2: there is k, with $1 \leq k \leq d$, such that $T_{i_1} = T_{i_2} = \cdots = T_{i_k}$ in Section 3. In Section 4, we will employ the result obtained in Section 3 to the special case in which the space is the torus $\mathbb{T}^d$, and transformations $R_{\alpha_1}, \dots, R_{\alpha_d} : \mathbb{T} \to \mathbb{T}$, satisfying that $1, \alpha_1, \cdots, \alpha_d$ are rationally independent. In Section 5, we give several examples to show that each alternative in Section 3 and Section 4 really occurs.

2. Preliminary

Let us first recall from [13, 17] some basic facts on measure theory and ergodic theory.

2.1. Measure Theory

In this section, X will be an arbitrary measure space equipped a positive measure μ.

Definition 2.1. [13] Let μ be a positive measure on X. A sequence $\{f_n\}$ of measurable functions on X is said to convergence in measure to the measurable function f if for every $\epsilon > 0$ there corresponds an N such that

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \epsilon \tag{2.1}$$