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The Approach of Solutions for the Nonlocal Diffusion Equation to Traveling Fronts

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Abstract The paper is concerned with the asymptotic behavior as $t \to \pm\infty$ of an entire solution $u(x,t)$ for the nonlocal diffusion equation. With bistable assumption, it is well known that the model has three different types of traveling fronts. Under certain conditions on the wave speeds, and by some auxiliary rational functions with certain properties to construct appropriate super- and sub solutions of the model, we establish two new types of entire solutions $u(x,t)$ which approach to three travelling fronts or the positive equilibrium as $t \to \pm\infty$.

Keywords Entire solution, Traveling front, Nonlocal evolution equation, Super-sub solutions.

MSC(2010) 34K05, 34A34, 34E05

1. Introduction

Consider the following nonlocal evolution problem for an entire solution $u(x,t)$ defined on $\mathbb{R} \times \mathbb{R}$:

$$u_t(x,t) = (J * u)(x,t) - u(x,t) + f(u(x,t)), \tag{1.1}$$

where the kernel J of the convolution $(J * u)(x,t) := \int_{\mathbb{R}} J(x-y)u(y,t)dy$ is nonnegative, even, with unit integral, and the function f is bistable. We see (1.1) is a nonlocal analog of the usual bistable reaction diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + f(u(x,t)). \tag{1.2}$$

As such, (1.1) as well as this equation (1.2) may model a variety of physical and biological phenomena involving media with properties varying in space. The possible advantages of (1.1) lie in the fact that much more general types of interactions between states at nearby locations in the medium can be accounted for. Lee et al. also [14] argue that, for processes where the spatial scale for movement is large in comparison with its temporal scale, nonlocal models may allow for better estimation of parameters from data and provide more insight into the biological system.

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Convolution equations are related to the classic Laplacian diffusion equations, i.e., letting $J(x) = \delta(x) + \delta''(x)$, where δ is the Dirac delta (see, Medlock et al. [16]), (1.2) reduces to (1.1).

It is well known that travelling fronts are special examples of the so-called entire solutions which are defined in the whole space and for all time $t \in \mathbb{R}$. Travelling fronts are known to exist for (1.1) with reaction terms f of bistable, monostable, and ignition type. For (1.1) with the bistable or ignition nonlinearities, the existence results of travelling fronts can be studied by Bates et al. [1], Chen [4], Coville [7] and Yagisita [24]. In the case where f is monostable, this kind of equation (1.1) was originally introduced in 1937 by Kolmogorov, Petrovskii and Piskunov [17] as a way to derive the Fisher equation (1.2) with $f(u) = u(1-u)$. We refer to some works [2,3,8,9,16,23] for (1.1) with the monostable nonlinearity. Notice that the entire solutions can help us for the mathematical understanding of transient dynamics. In recent years, there have been lots of works devoted to the interaction of traveling fronts and entire solutions for various diffusion equations; see, e.g., [11–13,15,18–20,22,25] and the references cited therein.

We point out that authors [18,19] investigated the entire solution behaving as two traveling fronts coming from both sides of the x-axis for (1.1) with the monostable and bistable nonlinearities, respectively. Recently, new types of entire solutions merging three fronts are concerned for some evolution equations with the bistable nonlinearity (see [5,6,21]). Motivated by these works, it is natural and interesting to study new entire solutions merging three fronts of (1.1) with the bistable nonlinearity.

Before to state our main results, we first give some assumptions for the functions $J(\cdot)$ and $f(\cdot)$, definitions of the traveling fronts and entire solutions for (1.1).

(J) $J(\cdot) \in C^1(\mathbb{R})$ has compact support, $J(x) = J(-x) \geq 0$,

$$\int_{\mathbb{R}} J(x)dx = 1 \text{ and } \int_{\mathbb{R}} J(x)e^{-\lambda x}dx < +\infty \text{ for all } \lambda > 0.$$

(A1) **[Bistable condition]** $f(u) \in C^2(\mathbb{R})$, $f(0) = f(a) = f(1) = 0$, $f'(0) < 0$, $f'(1) < 0$, $f'(a) > 0$, $f(u) < 0$ for any $u \in (0,a)$ and $f(u) > 0$ for any $u \in (a,1)$.

Definition 1.1 (traveling fronts and entire solutions)**.**

(1) A solution $u(x,t)$ is called a traveling front of (1.1) connecting $\{e_1, e_2\} \subset \{0, a, 1\}$ with the wave speed k, if

$$u(x,t) = \phi(x+kt), \text{ or } u(x,t) = \phi(-x+kt),$$

for all $x \in \mathbb{R},\ t \in \mathbb{R}$ and some function $\phi(\cdot) = \{\phi(\cdot)\}_{x\in\mathbb{R}, t\in\mathbb{R}}$, which satisfies

$$\phi(-\infty) = e_1 \text{ and } \phi(+\infty) = e_2, \quad \forall x \in \mathbb{R}.$$

(2) A function $u(x,t) = \{u(x,t)\}_{x\in\mathbb{R},t\in\mathbb{R}}$ is called an entire solution of (1.1) if for any $x \in \mathbb{R}$, $u(x,t)$ is differentiable for all $t \in \mathbb{R}$ and satisfies (1.1) for $x \in \mathbb{R}$ and $t \in \mathbb{R}$.

Now we recall the result in [1,4,24] as follows.