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Existence and Multiplicity of Positive Solutions for Fractional Differential Equation with Parameter*

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Abstract In this paper, by using the fixed point theorem for a cone map, we study the existence and multiplicity of positive solutions for a class of fractional differential equation with parameter.

Keywords Fractional differential equation, Green function, Cone.

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1. Introduction

Fractional calculus has played a significant role in engineering, science, economy, and other fields, during the last few decades. There has been a significant development in ordinary and partial differential equations involving fractional derivatives. There are many important results about the existence of solutions for fractional differential equation, see [1, 2, 5–8, 10, 11] for more details.

In [1], Bai and Lü considered the positive solutions for boundary value problem of fractional order differential equation

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t,\ u(t)) = 0,\ t \in (0,\ 1), \\ u(0) = 0, u(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2, f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

In this paper, under different growth conditions of f, we obtain the existence and multiplicity of positive solutions for boundary value problem of fractional differential equation with parameter

$$\begin{cases} D^{\alpha}_{0+}u(t) + \lambda f(t,\ u(t)) = 0,\ t \in (0,\ 1), \\ u(0) = 0, u(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2, f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous, $\lambda > 0$ is a parameter.

Remark 1.1. When $\alpha = 2$, problem (1.1) is reduced to the problem of paper [4].

We make the following hypotheses:

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(A) $f(t,u) : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and there exists $g \in C((0,+\infty),(0,+\infty)), q_1, q_2 \in C((0,1),(0,+\infty))$ such that $q_1(t)g(y) \leq f(t, t^{\alpha-2}y) \leq q_2(t)g(y), t \in [0,1], y \in [0,+\infty)$.

For the convenience, we take some notations. Let

$$g_0 = \lim_{y\to 0^+} \frac{g(y)}{y}, \qquad g_\infty = \lim_{y\to+\infty} \frac{g(y)}{y}.$$

i_0=numbers of zeros in the set $\{g_0, g_\infty\}$; i_∞=numbers of infinities in the set $\{g_0, g_\infty\}$.

$$M(p) = \max_{0\leq y\leq p}\{g(y)\}, \qquad m(p) = \min_{\frac{(\alpha-1)p}{16}\leq y\leq p}\{g(y)\}.$$

The tool theorem is following:

Theorem 1.1. *Let E be a Banach space, $K \subset E$ is a cone, Ω_1, Ω_2 are bounded open subsets of E, $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, suppose that $A : K\bigcap(\overline{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous and satisfies :*

(i) $\|Ax\| \leq \|x\|, x \in K\bigcap\partial\Omega_1$, and $\|Ax\| \geq \|x\|, x \in K\bigcap\partial\Omega_2$; or

(ii) $\|Ax\| \geq \|x\|, x \in K\bigcap\partial\Omega_1$, and $\|Ax\| \leq \|x\|, x \in K\bigcap\partial\Omega_2$;

Then A has a fixed point in $K\bigcap(\overline{\Omega}_2 \setminus \Omega_1)$.

Definition 1.1. We call $D^\alpha_{0+}f(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt, \alpha > 0, n = [\alpha]+1$ is the Riemann-Liouville fractional derivative of order α.$[\alpha]$ denotes the integer part of number α.

Definition 1.2. We call $I^\alpha_{0+}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt, x > 0, \alpha > 0$ is Riemann-Liouville fractional integral of order α.

2. Preliminaries

Lemma 2.1 (Lemma 2.3, [1]). *The solutions of problem*

$$\begin{cases} D^\alpha_{0+}u(t) + \lambda f(t,\ u(t)) = 0,\ t \in (0,\ 1), \\ u(0) = 0, u(1) = 0 \end{cases}$$

is equivalent to the solutions of the integral equation

$$u(t) = \lambda\int_0^1 G(t,s)f(s,u(s))ds, \tag{2.1}$$

where

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)},\ 0 \leq s \leq t \leq 1; \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, \qquad 0 \leq t \leq s \leq 1. \end{cases}$$

Lemma 2.2 (Proposition 1, [5]). *The Green function $G(t,s)$ has the following properties:*

(i) $G(t,s) \in C([0,1]\times[0,1])$ and $G(t,s) > 0, \forall t,s \in (0,1)$;