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Mixed Boundary Value Problems for a Class of Fractional Differential Equations with Impulses*

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Abstract We investigate a class of mixed boundary value problem of nonlinear impulsive fractional differential equations with a parameter. The uniqueness of this problem is proved by applying Banach fixed point theorem.

Keywords Fractional differential equations, Boundary value problem, Impulsive, Uniqueness.

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1. Introduction

Fractional differential equation can depict the phenomenon, dynamic procedure, electroanalytical chemistry as well as control theory, signal processing in our life more perfectly [6, 8, 10]. Boundary value problem of fractional differential equation has been studied by many scholars [1, 4, 5, 11–13, 15–18]. The research of impulsive differential equations began in 20th century, through researchers cooperation, the basic theory of existence of solution for impulsive differential equations had been set up [3, 9], many essential results have also been obtained [7, 14].

In [14], Guo studied the hybrid boundary value problem of fractional differential equations with impulse

$$\begin{cases} D^\alpha u(t) = f\big(t, u(t)\big), \quad 0 < t < T, 1 < \alpha \le 2, \\ \triangle u(t_k) = I_k\big(u(t_k)\big), \triangle u'(t_k) = I_k^*\big(u(t_k)\big), k = 1, 2, \cdots, p, \\ Tu'(0) = -au(0) - bu(T), Tu'(T) = cu(0) + du(T), a, b, c, d \in R. \end{cases}$$

where D^α is the Caputo derivative, $1 < \alpha \le 2$, $J = [0, T]$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = T$, $J' = J \backslash \{t_1, \cdots, t_p\}$, $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, $\triangle u'(t_k) = u'(t_k^+) - u'(t_k^-)$. By using Leray-schauder fixed point theorem, the author obtained the existence and uniqueness of this problem.

Motivated by the above work, in this paper, we consider the uniqueness of the solution for a class of mixed boundary value problem of nonlinear impulsive

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differential equations of order $\alpha \in (2,3]$ given by

$$\begin{cases} D^{\alpha}u(t) = \lambda f\big(t, u(t)\big), \quad t \in J', \\ \triangle u(t_k) = Q_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \triangle u'(t_k) = R_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \triangle u''(t_k) = S_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, u''(0) + u''(1) = 0, \end{cases} \tag{1.1}$$

where D^{α} is the Caputo derivative, $2 < \alpha \leq 3$, $J = [0,1]$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $Q_k, R_k, S_k \in C(\mathbb{R}, \mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1$, $J' = J \backslash \{t_1, \cdots, t_m\}$, $J_0 = [0, t_1], J_1 = (t_1, t_2], \cdots, J_m = (t_m, 1]$, $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, $\triangle u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $\triangle u''(t_k) = u''(t_k^+) - u''(t_k^-)$.

The paper is organized as follows. In Section 2, we introduce some necessary notions, basic definitions and Lemmas. In Section 3, by using Banach fixed point theorem, the uniqueness of solutions is proved. In Section 4, we give an example to demonstrate the applications of our results.

2. Preliminaries

Definition 2.1. ([10]) The Caputo derivative of order $\alpha > 0$ of a continuous function $u : J \to \mathbb{R}$ is given by

$$D_t^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, n = [\alpha] + 1, \alpha > 0.$$

Lemma 2.1. *([10]) Let (X, d) be a complete metric space, and $\overline{\Omega}$ be a convex closed subset of X. $T : \overline{\Omega} \to \overline{\Omega}$ is a contraction mapping:*

$$d(Tx, Ty) \leq kd(x, y),$$

where $0 < k < 1$, for each $x, y \in \overline{\Omega}$. Then, there exists a unique fixed point x of T in $\overline{\Omega}$, i.e. $Tx = x$.

Lemma 2.2. *([2]) Let $u(t) \in C[0,1]$, and $\alpha > 0$, the solution of the fractional differential equation*

$$D^{\alpha}u(t) = 0$$

is $u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1}, C_i \in \mathbb{R}, i = 0, 1, 2, \cdots, n, n = [\alpha] + 1$.

Lemma 2.3. *For a given $f \in C(J \times \mathbb{R}, \mathbb{R})$, a function u is a solution of the following impulsive boundary value problem*

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t)), 2 < \alpha \leq 3, \quad t \in J', \\ \triangle u(t_k) = Q_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \triangle u'(t_k) = R_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \triangle u''(t_k) = S_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, u''(0) + u''(1) = 0, \end{cases} \tag{2.1}$$