Journal of Nonlinear Modeling and Analysis
Volume 3, Number 1, March 2021, 53–70
http://jnma.ca; http://jnma.ijournal.cn
DOI:/jnma.2021.53

Existence of Periodic Solutions in Impulsive Differential Equations*

Fangfang Jiang[1,†]

Abstract In this paper, we are concerned with the problem of existence of periodic solutions for a class of second order impulsive differential equations. By Poincaré-Bohl theorem, we give several criteria to guarantee that the impulsive differential equation has periodic solutions under assumptions that the nonlinear term satisfies the linear growth conditions. Two specific examples are presented to illustrate the obtained results.

Keywords Impulse, Poincaré-Bohl theorem, Periodic solution, Existence.

MSC(2010) 34A37, 34C25, 34A12.

1. Introduction

In this paper, we consider the following second order impulsive differential equation

$$\begin{cases} x'' + g(x) = e(t,x,x'), & t \neq t_k,\ t \in \mathbb{R}, \\ x(t_k^+) = a_k x(t_k), & \\ x'(t_k^+) = b_k x'(t_k), & t = t_k,\ k \in \mathbb{Z}_+, \end{cases} \tag{1.1}$$

where $(x(t_k), x'(t_k)) = (x(t_k^-), x'(t_k^-))$, $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying the linear growth condition

$$0 < l \leq \lim_{|x| \to +\infty} \frac{g(x)}{x} \leq L < +\infty,$$

$e : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, bounded and 2π-periodic to the first variable, $0 < t_k < t_{k+1} \uparrow +\infty$, $a_k > 0, b_k > 0$ are constants and there exists a positive integer q such that $a_{k+q} = a_k, b_{k+q} = b_k$ and $t_{k+q} = t_k + 2\pi$ for $k \in \mathbb{Z}_+$, $\mathbb{Z}_+$ is the set of positive integers.

Impulsive effects widely exist in many evolution processes, in which their states are changed abruptly at some moments. Impulsive differential equation has been developed by many mathematicians. Please see the classical monographs [1, 13], and [8, 9, 15–19, 21–24] for the existence of periodic solutions. In addition, applications of the impulsive differential equation with/or without delays occur in biology, mechanics, engineering etc., see for example [11, 25–27] and the references therein.

† the corresponding author.
Email address: jiangfangfang87@126.com(F. Jiang)
[1] School of Science, Jiangnan University, Wuxi, Jiangsu 214122, China

* The work is supported by National Natural Science Foundation of China (11701224) and Natural Science Foundation of Jiangsu Provincial Youth Foundation (BK20170168).

The continuous case of (1.1) without impulses is as follows

$$x'' + g(x) = e(t, x, x'). \tag{1.2}$$

This type of second order differential equation is one of the typical models both in ODE and forced vibrations. Particularly, the Duffing equation ($i.e.$ $x''+g(x) = e(t)$) is a class of mathematical and physical equations, it has many important applications in mechanical and electrical engineering. Recent decades, the existence of periodic solutions has been extensively studied for the Duffing equation under assumptions that the nonlinear term g satisfies superlinear, sublinear or semilinear conditions. The methods include fixed point theorem, variational method and topological degree theory [2–7, 10, 14].

When $e(t, x, x') = e(t)$, (1.2) is conservative. The Poincaré mapping is an area-preserving homeomorphism. By Poincaré-Birkhoff theorem, Jacobowitz [10] gave the first application to periodic solutions of second order differential equations, and proved the existence of infinitely many periodic solutions. See for example [4–6, 20] for some related researches and the references therein. Recently, the Poincaré-Birkhoff theorem has been applied to the impulsive differential equation [12, 17, 18]. In [12], Jiang et al. gave the first application of the Poincaré-Birkhoff theorem to the following impulsive Duffing equations at resonance

$$\begin{cases} x'' + g(x) = e(t), & t \neq t_k, \\ x(t_k^+) = a_k x(t_k), & \\ x'(t_k^+) = b_k x'(t_k), & k \in \mathbb{Z}_+, \end{cases} \tag{1.3}$$

under the superlinear condition. By using a method of comparing rotational inertia, the authors obtained the multiplicity of periodic solutions. In [18], Qian et al. used a geometric method to study the periodic solutions for a superlinear second order differential equation with general impulsive effects as follows

$$\begin{cases} x'' + g(x) = e(t, x, x'), & t \neq t_k, \\ \triangle x(t_k) = I_k(x(t_k), x'(t_k)), & \\ \triangle x'(t_k) = J_k(x(t_k), x'(t_k)), & k = \pm 1, \pm 2, \cdots, \end{cases} \tag{1.4}$$

and obtained the multiplicity when (1.4) is conservative. In [17], Niu and Li studied a conservative semilinear impulsive Duffing equation, in which their states occur one jump only in $[0, 2\pi]$. Similarly, by the Poincaré-Birkhoff theorem they proved the existence of infinitely many periodic solutions for autonomous and nonautonomous equations respectively. On the other hand, when (1.2) is nonconservative, Ding [7] developed a new twist fixed point theorem used for nonarea-preserving mappings. In [18], Qian et al. further studied the existence of periodic solutions for (1.4) being nonconservative.

In this paper, we study the existence of periodic solutions for a class of semilinear second order impulsive differential equations (1.1). Our aim is to estimate the time that any solution trajectory of the system rotates one circle on the phase plane. By using the Poincaré-Bohl theorem to obtain the existence results. As far as I know, there is no result on the existence of periodic solutions for the semilinear second order differential equations with the linear impulsive effects.

Now we recall an existence result from the Poincaré-Bohl theorem [4].