Journal of Nonlinear Modeling and Analysis
Volume 3, Number 1, March 2021, 13–34
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2021.13

The Maximum Number of Zeros of Functions with Parameters and Application to Differential Equations*

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Abstract In this paper, we first study the problem of finding the maximum number of zeros of functions with parameters and then apply the results obtained to smooth or piecewise smooth planar autonomous systems and scalar periodic equations to study the number of limit cycles or periodic solutions, improving some fundamental results both on the maximum number of limit cycles bifurcating from an elementary focus of order k or a limit cycle of multiplicity k, or from a period annulus, and on the maximum number of periodic solutions for scalar periodic smooth or piecewise smooth equations as well.

Keywords Maximum number, Multiplicity, Limit cycle, Piecewise smooth equation.

MSC(2010) 34C07, 34C23.

1. Introduction

As we know, in the study of differential equations a very important aspect is the number of limit cycles for planar systems or the number of periodic solutions for scalar periodic equations. There have been certain classical and fundamental theorems on this aspect. For example, in a family of C^∞ systems with parameters a limit cycle of multiplicity k generates at most k limit cycles, and an elementary focus of order k generates at most k limit cycles. In a family of C^∞ near-Hamiltonian systems with parameters, the total number of the first order Melnikov function can control the number of limit cycles bifurcating from a period annulus. These important results have many applications to polynomial systems. The proofs of the results are all very similar with two main steps. The first step is to establish a suitable bifurcation function. The second step is to analyze the number of zeros of the bifurcation function by using the method of contradiction. See Theorems 1.3.2,

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*The first author was supported by National Natural Science Foundation of China (11771296, 11931016). The second author was supported by National Natural Science Foundation of China (11971145) and Natural Science Foundation of Hebei Province(A2019205133).

2.3.2 and 3.1.4 in Han [5], Theorem 2.4 in Part II of Christopher and Li [2] and Theorem 1.1 in Han [6]. However, from the proofs of these theorems, the multiplicity of the limit cycles bifurcated is not considered.

The aim of this paper is to further study the problem of finding the maximum number of zeros of functions with parameters and then improve some fundamental results on the maximum number of limit cycles bifurcating from an elementary focus of order k or a limit cycle of multiplicity k, or from a period annulus. We study the same problem for piecewise smooth systems on the plane and scalar periodic smooth or piecewise smooth equations as well. As a preliminary, we first study the maximum number of zeros of functions with parameters in Section 2. Then, based on the main results of Section 2, we study the maximum number of limit cycles in planar autonomous systems (smooth or piecewise smooth) or of periodic solutions in scalar periodic equations (smooth or piecewise smooth) in the rest sections.

2. The number of zeros of functions with parameters

Let $F : I \times D \to \mathbf{R}$ be a C^k function, where $I \subset \mathbf{R}$ is an open interval, $D = \{\lambda \| \lambda| < \varepsilon\} \subset \mathbf{R}^n, \varepsilon > 0, k \geq 1, n \geq 1$. As we know, for a fixed $\lambda \in D$ and an integer $1 \leq l \leq k$ we say that $x_0 \in I$ is a zero of F in x with multiplicity l if

$$\frac{\partial^l F}{\partial x^l}(x_0, \lambda) \neq 0,\ \frac{\partial^j F}{\partial x^j}(x_0, \lambda) = 0,\ j = 0, 1, \cdots, l-1.$$

In this section, we study the number of zeros of the function $F(x, \lambda)$ in x based on the number of zeros of the unperturbed function $F(x, 0) \equiv f(x)$. For the purpose, we need to present or establish some preparation lemmas.

First, from Section 1.3 of [7], we have the following lemma.

Lemma 2.1. *Let $x_0 \in I$, $1 \leq l \leq k$. Then, there exists a C^{k-l} function $\bar{R} : I \times D \to \mathbf{R}$ such that for $(x, \lambda) \in I \times D$*

$$F(x, \lambda) = \sum_{j=0}^{l-1} \frac{1}{j!} \frac{\partial^j F}{\partial x^j}(x_0, \lambda)(x - x_0)^j + (x - x_0)^l \bar{R}(x, \lambda)$$

with

$$\bar{R}(x_0, \lambda) = \frac{1}{l!} \frac{\partial^l F}{\partial x^l}(x_0, \lambda).$$

If $F \in C^\infty(I \times D)$, then $\bar{R} \in C^\infty(I \times D)$ for any given $l \geq 1$.

The conclusion of the above lemma is a simple improvement of the well-known Taylor formula. The key point is the smoothness of $\bar{R}$.

The following lemma looks obvious and its proof needs to use Lemma 2.1.

Lemma 2.2. *Let $x_0 \in I$, $1 \leq l \leq k$. Then, $x = x_0$ is a zero of the unperturbed function f with multiplicity l if and only if there exists a C^{k-l} function $g : I \to \mathbf{R}$ satisfying $g(x_0) \neq 0$ such that*

$$f(x) = (x - x_0)^l g(x),\ x \in I.$$