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Finitely Generated Torsion-free Nilpotent Groups Admitting an Automorphism of Prime Order

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Abstract: Let G be a finitely generated torsion-free nilpotent group and α an automorphism of prime order p of G. If the map $\varphi : G \longrightarrow G$ defined by $g^{\varphi} = [g,\ \alpha]$ is surjective, then the nilpotent class of G is at most $h(p)$, where $h(p)$ is a function depending only on p. In particular, if $\alpha^3 = 1$, then the nilpotent class of G is at most 2.

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1 Introduction and Main Results

An automorphism of a group G is called regular if it moves every element of G except the identity. Burnside[1] proved the following classical result.

Proposition 1.1 *A finite group G admits a regular automorphism α of order 2 if and only if G is abelian of odd order.*

In Proposition 1.1, the condition on the finite groups is essential because Proposition 1.1 is incorrect for infinite groups. For example, let F be a free group generated by x and y, the automorphism α defined by $x^{\alpha} = y$ and $y^{\alpha} = x$ is regular of order 2. But F is not an abelian group.

In Proposition 1.1, if α is an arbitrary automorphism of order 2 of a group G and the map $\varphi : G \longrightarrow G$ defined by $g^{\varphi} = [g,\ \alpha]$ is surjective, we obtain the following proposition.

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Proposition 1.2 *Let G be a group and α an automorphism of order 2 of G. If the map $\varphi : G \longrightarrow G$ $(g \longmapsto [g,\ \alpha])$ is surjective, then G is abelian.*

Proof. Since φ is surjective, for any $x \in G$, there exists some $g \in G$ such that

$$x = [g,\ \alpha] = g^{-1}g^{\alpha}.$$

Moreover,

$$x^{\alpha} = (g^{-1}g^{\alpha})^{\alpha} = (g^{-1})^{\alpha}g^{\alpha^2} = (g^{\alpha})^{-1}g = x^{-1}.$$

Thus, for any $g_1, g_2 \in G$, we have

$$(g_1^{-1}g_2^{-1})^{\alpha} = (g_1^{-1}g_2^{-1})^{-1} = g_2g_1$$

and

$$(g_1^{-1}g_2^{-1})^{\alpha} = (g_1^{-1})^{\alpha}(g_2^{-1})^{\alpha} = g_1g_2.$$

Obviously, $g_1g_2 = g_2g_1$. Hence G is abelian. This completes the proof.

For a regular automorphism of order 3 of an arbitrary group, Neumann[2] proved the following result.

Proposition 1.3 *Let G be a group and α a regular automorphism of order 3 of G. If the map $\varphi : G \longrightarrow G$ $(g \longmapsto [g,\ \alpha])$ is surjective, then G is nilpotent of class at most 2.*

For a regular automorphism of prime order of a finite group, Thompson[3] proved that if a finite group G has a regular automorphism of prime order, then G is nilpotent. For deeper results concerning regular automorphisms see [4].

Abandoning the condition on regularity, we are interested in the arbitrary automorphism of prime order of a group. In this paper, we study the arbitrary automorphism of order 3 of a finitely generated torsion-free nilpotent group, and obtain the following result which generalizes the above result of Neumann[2].

Theorem 1.1 *Let G be a finitely generated torsion-free nilpotent group and α an automorphism of order 3 of G. If the map $\varphi : G \longrightarrow G$ $(g \longmapsto [g,\ \alpha])$ is surjective, then the nilpotent class of G is at most 2.*

Furthermore, we consider the automorphism of prime order p of a finitely generated torsion-free nilpotent group.

Theorem 1.2 *Let G be a finitely generated torsion-free nilpotent group and α an automorphism of prime order p of G. If the map $\varphi : G \longrightarrow G$ $(g \longmapsto [g,\ \alpha])$ is surjective, then the nilpotent class of G is at most $h(p)$, where $h(p)$ is a function depending only on p.*

2 Proof of Theorem 1.1

Lemma 2.1 *Let G be a group and α an automorphism of order 3 of G. If the map $\varphi : G \longrightarrow G$ $(g \longmapsto [g,\ \alpha])$ is surjective, then for any $x \in G$, we have $xx^{\alpha}x^{\alpha^2} = 1$.*