Ann. Appl. Math.
doi: 10.4208/aam.OA-2020-0002

Vol. **37**, No. 1, pp. 22-30
February 2021

A Simple Proof of Regularity for $C^{1,\alpha}$ Interface Transmission Problems

Hongjie Dong*

Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA

Received 27 November 2020; Accepted (in revised version) 5 February 2021

Abstract. We give an alternative proof of a recent result in [1] by Caffarelli, Soria-Carro, and Stinga about the $C^{1,\alpha}$ regularity of weak solutions to transmission problems with $C^{1,\alpha}$ interfaces. Our proof does not use the mean value property or the maximum principle, and also works for more general elliptic systems with variable coefficients. This answers a question raised in [1]. Some extensions to $C^{1,\mathrm{Dini}}$ interfaces and to domains with multiple sub-domains are also discussed.

AMS subject classifications: 35B65, 35B45, 35J47, 35J25

Key words: Transmission problems, elliptic systems with variable coefficients, $C^{1,\alpha}$ and $C^{1,\mathrm{Dini}}$ interfacial boundaries, C^{α} and Dini coefficients.

---

1 Introduction and main results

In a recent paper [1], Caffarelli, Soria-Carro, and Stinga studied the following transmission problem. Let $\Omega \in \mathbb{R}^d$ be a smooth bounded domain with $d \geq 2$, and Ω_1 be a sub-domain of Ω such that $\Omega_1 \subset\subset \Omega$ and $\Omega_2 = \Omega \setminus \overline{\Omega_1}$. Assume that the interfacial boundary $\Gamma(=\partial\Omega_1)$ between Ω_1 and Ω_2 is $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Consider the elliptic problem with the transmission conditions

$$\begin{cases} \Delta u = 0 & \text{in } \ \Omega_1 \cup \Omega_2, \\ u = 0 & \text{on } \ \partial\Omega, \\ u|_\Gamma^+ = u|_\Gamma^-, \quad \partial_\nu u|_\Gamma^+ - \partial_\nu u|_\Gamma^- = g, \end{cases} \tag{1.1}$$

*Corresponding author. *Email address:* Hongjie_Dong@brown.edu (H. Dong)

where g is a given function on Γ, ν is the unit normal vector on Γ which is pointing inside Ω_1, and $u|_\Gamma^+$ and $u|_\Gamma^-$ (and $\partial_\nu u|_\Gamma^+$ and $\partial_\nu u|_\Gamma^-$) are the left and right limit of u (and its normal derivative, respectively) on Γ in Ω_1 and Ω_2. The main result of [1] can be formulated as the following theorem.

Theorem 1.1. *Under the assumptions above, for any $g\in C^\alpha(\Gamma)$, there is a unique weak solution $u\in H^1(\Omega)$ to* (1.1)*, which is piecewise $C^{1,\alpha}$ up to the boundary in Ω_1 and Ω_2 and satisfies*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega_1})}+\|u\|_{C^{1,\alpha}(\overline{\Omega_2})}\le N\|g\|_{C^\alpha(\Gamma)},$$

where $N=N(d,\alpha,\Omega,\Gamma)>0$ is a constant.

The proof in [1] uses the mean value property for harmonic functions and the maximum principle together with an approximation argument. We refer the reader to [1] for earlier results about the transmission problem with smooth interfacial boundaries. The main feature of Theorem 1.1 is that Γ is only assumed to be in $C^{1,\alpha}$, which is weaker than those in the literature. In Remark 4.5 of [1], the authors raised the question of transmission problems with variable coefficient operator and mentioned two main difficulties in carrying over their proof to the general case.

In this paper, we answer this question by giving a alternative proof of Theorem 1.1, which does not invoke the mean value property or the maximum principle, and also works for more general non-homogeneous elliptic systems in the form

$$\begin{cases} \mathcal{L}u:=D_k(A^{kl}D_lu)=\operatorname{div}F+f & \text{in } \Omega_1\cup\Omega_2,\\ u=0 & \text{on } \partial\Omega,\\ u|_\Gamma^+=u|_\Gamma^-, \quad A^{kl}D_lu\nu_k|_\Gamma^+-A^{kl}D_lu\nu_k|_\Gamma^-=g, \end{cases} \tag{1.2}$$

where the Einstein summation convention in repeated indices is used,

$$u=(u^1,\cdots,u^n)^\top, \quad F_k=(F_k^1,\cdots,F_k^n)^\top, \quad f=(f^1,\cdots,f^n)^\top, \quad g=(g^1,\cdots,g^n)^\top,$$

are (column) vector-valued functions, for k, $l=1,\cdots,d$, $A^{kl}=A^{kl}(x)$ are $n\times n$ matrices, which are bounded and satisfy the strong ellipticity with ellipticity constant $\kappa>0$:

$$\kappa|\xi|^2\le A_{ij}^{kl}\xi_k^i\xi_l^j, \qquad |A^{kl}|\le\kappa^{-1},$$

for any $\xi=(\xi_k^i)\in\mathbb{R}^{n\times d}$.

Theorem 1.2. *Assume that Ω_1, Ω_2, and Γ satisfy the conditions in Theorem* 1.1*, A^{kl} and F are piecewise C^α in Ω_1 and Ω_2, $g\in C^\alpha(\Gamma)$, and $f\in L_\infty(\Omega)$. Then there is a*