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Remarks on Blow-Up Phenomena in p-Laplacian Heat Equation with Inhomogeneous Nonlinearity

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Abstract. We investigate the p-Laplace heat equation $u_t - \Delta_p u = \zeta(t) f(u)$ in a bounded smooth domain. Using differential-inequality arguments, we prove blow-up results under suitable conditions on ζ, f, and the initial datum u_0. We also give an upper bound for the blow-up time in each case.

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1 Introduction

In the past decade a strong interest in the phenomenon of blow-up of solutions to various classes of nonlinear parabolic problems has been assiduously investigated. We refer the reader to the books [1, 2] as well as to the survey paper [3]. Problems with constant coefficients were investigated in [4], and problems with time-dependent coefficients under homogeneous Dirichlet boundary conditions were treated in [5]. See also [6] for a related system. The question of blow-up for nonnegative classical solutions of p-Laplacian heat equations with various boundary conditions has attracted considerable attention in the mathematical community in recent years. See for instance [7–10].

There are two effective techniques which have been employed to prove non-existence of global solutions: the concavity method ([11]) and the eigenfunction method ([12]). The

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latter one was first used for bounded domains but it can be adapted to the whole space $\mathbb{R}^N$. The concavity method and its variants were used in the study of many nonlinear evolution partial differential equations (see, e.g., [13–15]).

In the present paper, we investigate the blow-up phenomena of solutions to the following nonlinear p-Laplacian heat equation:

$$\begin{cases} u_t-\Delta_p u=\zeta(t)f(u), & x\in\Omega, \quad t>0, \\ u(t,x)=0, & x\in\partial\Omega, \quad t>0, \\ u(0,x)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u:=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian operator, $p\geq 2$, Ω is a bounded sufficiently smooth domain in $\mathbb{R}^N$, $\zeta(t)$ is a nonnegative continuous function. The nonlinearity $f(u)$ is assumed to be continuous with $f(0)=0$. More specific assumptions on f, ζ and u_0 will be made later.

The case of $p=2$ has been studied in [4] for $\zeta(t)\equiv 1$, and in [5] for ζ being a nonconstant function of t. Concerning the case $p>2$, Messaoudi [10] proved the blow-up of solutions with vanishing initial energy when $\zeta(t)\equiv 1$. See also [9] and references therein. Recently, a p-Laplacian heat equations with nonlinear boundary conditions and time-dependent coefficient was investigated in [7]. This note may be regarded as a complement, and in some sense an improvement, of [5, 10].

Let us now precise the assumptions on f and ζ. If $p=2$, we suppose either

$$f\in C^1(\mathbb{R}) \quad \text{is convex with} \quad f(0)=0; \tag{1.2}$$

$$\exists\ \lambda>0 \quad \text{such that} \quad f(s)>0 \quad \text{for all} \quad s\geq\lambda; \tag{1.3}$$

$$\int_\lambda^\infty \frac{\mathrm{d}s}{f(s)}<\infty; \tag{1.4}$$

$$\inf_{t\geq 0}\left(\int_0^t (\zeta(s)-1)\,\mathrm{d}s\right):=m\in(-\infty,0], \tag{1.5}$$

or

$$sf(s)\geq(2+\epsilon)F(s)\geq C_0|s|^\alpha, \tag{1.6}$$

for some constants $\epsilon, C_0>0$, $\alpha>2$, and

$$\zeta\in C^1([0,\infty)) \quad \text{with} \quad \zeta(0)>0 \text{ and } \zeta'\geq 0. \tag{1.7}$$

Here $F(s)=\displaystyle\int_0^s f(\tau)\,\mathrm{d}\tau$.

Our first main result concerns the case $p=2$ and reads as follows.

Theorem 1.1. *Suppose that assumptions* (1.2)–(1.5) *are fulfilled. Let* $0\leq u_0\in L^\infty(\Omega)$ *such that* $\displaystyle\int_\Omega u_0\phi_1$ *is large enough. Then the solution* $u(t,x)$ *of problem* (1.1) *blows up in finite time.*