

Localized Exponential Time Differencing Method for Shallow Water Equations: Algorithms and Numerical Study

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Abstract. In this paper, we investigate the performance of the exponential time differencing (ETD) method applied to the rotating shallow water equations. Comparing with explicit time stepping of the same order accuracy in time, the ETD algorithms could reduce the computational time in many cases by allowing the use of large time step sizes while still maintaining numerical stability. To accelerate the ETD simulations, we propose a localized approach that synthesizes the ETD method and overlapping domain decomposition. By dividing the original problem into many subdomain problems of smaller sizes and solving them locally, the proposed approach could speed up the calculation of matrix exponential vector products. Several standard test cases for shallow water equations of one or multiple layers are considered. The results show great potential of the localized ETD method for high-performance computing because each subdomain problem can be naturally solved in parallel at every time step.

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1 Introduction

The exponential time differencing (ETD) method, as an exponential integrator-based method, has been developed for solving evolutionary partial differential equations of semi-linear or fully nonlinear types (see, for example, [1–8] and a thorough review [9]). Such a method is constructed on the basis of exponential integrators and variation-of-constants formula, and is known for its desirable numerical stability. Indeed, for stiff problems, a large time step size can be used in ETD, while tiny time step sizes are often required by explicit time stepping. In addition, differently from standard implicit time stepping, nonlinear solvers are not required in ETD. Therefore, the ETD method usually leads to significant computational savings comparing with other time-stepping algorithms. The major computational efforts in ETD schemes are spent in evaluating matrix exponential and vector products. Many algorithms have been proposed in order to evaluate matrix exponentials [10–15]. Some of them are designed for large sparse matrices, while the others only work for matrices of moderate size. We are interested in the former as the discrete system we consider in this paper is of large dimensions. The first comprehensive package for evaluating large-scale matrix exponential vector products is EXPOKIT, which was developed by Sidje in [12]. The backbone of its sparse routines is Krylov subspace projection method such as the Arnoldi and Lanczos processes whose mathematical foundation was established in [16–18]. By projecting large matrices to smaller ones via the Krylov subspace approach, the corresponding matrix exponential becomes easier to compute. The other main idea in EXPOKIT is to adapt the time step size of the ETD simulations based on an error estimator developed in [17]. Combing the time stepping idea of EXPOKIT and the adaptivity of the dimension of the Krylov subspace [1], the *phipm* function algorithm was developed in [13]. This algorithm achieves a balance between the time stepping error and Krylov projection error by dynamically choosing the dimension of the Krylov subspace and the size of time stepping. But it was pointed out in [19] that the Arnoldi procedure still takes too much time, which made *phipm* less efficient than other semi-implicit predictor-corrector schemes for geophysical fluid dynamics problems. Thus, the Krylov subspace with the incomplete orthogonalization procedure (*phipm/IOM2*) was recently introduced, which has been successfully applied, in the context of exponential Rosenbrock integration methods, to the shallow water equations on the sphere [15]. Another solver, KIOPS, was proposed in [14] for calculating φ -functions in exponential integrators to allow efficient implementation of multi-stage exponential integrators.

To accelerate the exponential time integration, a different research direction is to take advantage of parallel and high-performance computing. A straightforward way is to perform the parallelization at the algebraic level. For instance, the parallel adaptive-Krylov exponential solver was proposed in [20], where the standard data-parallel approach is taken, that is, each vector is split across all the processors and MPI communication is used for performing the vector algebraic operations. However, since the matrix exponential is global and dense, this approach usually requires a high communication volume.