INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 6, Pages 872–899

A CLASS OF BUBBLE ENRICHED QUADRATIC FINITE VOLUME ELEMENT SCHEMES ON TRIANGULAR MESHES

YANHUI ZHOU

Abstract. In this work, we propose and analyze a class of bubble enriched quadratic finite volume element schemes for anisotropic diffusion problems on triangular meshes. The trial function space is defined as quadratic finite element space by adding a space which consists of element-wise bubble functions, and the test function space is the piecewise constant space. For the class of schemes, under the coercivity result, we proved that $|u - u_h|_1 = \mathcal{O}(h^2)$ and $\|u - u_h\|_0 = \mathcal{O}(h^3)$, where u is the exact solution and u_h is the bubble enriched quadratic finite volume element solution. The theoretical findings are validated by some numerical examples.

Key words. Bubble enriched quadratic finite volume element schemes, anisotropic diffusion problems, triangular meshes, H^1 and L^2 error estimates.

1. Introduction

Due to the local conservation law and other advantages, the finite volume method (FVM) is one of the most important numerical methods for solving partial differential equations, see e.g. [2, 21, 30, 31, 34]. The finite volume element (FVE) method (FVEM) is a special type of FVM, and attracted many researchers attention (e.g. [22, 25, 43]).

The coercivity result is a basis for the error estimate of FVEM. For the linear FVEM over triangular meshes, its element stiffness matrix can be regarded as a small perturbation of linear finite element method for variable coefficient, the coercivity result then follows (c.f. [1, 4, 16, 17, 38]). However, the coercivity analysis for the quadratic scheme is not easy. For instance, assume that the maximum angle of each triangular element is not greater than $90°$, and the ratio of the lengths of the two sides of the maximum angle belong to $[\sqrt{2/3}, \sqrt{3/2}]$, then Tian and Chen [33] presented a coercivity result for the first proposed quadratic scheme in 1991. In 1996, Liebau [24] proposed another quadratic scheme, and required that the geometry of the triangulation triangles is not too extreme. In 2009, Xu and Zou [38] introduced a general framework to construct quadratic schemes, and improved some coercivity results for the schemes presented in [15, 24, 33] . In 2012, Chen, Wu and Xu [11] presented a general framework for construction and analysis of higher-order FVMs, under their framework, the minimum angle conditions of the schemes in [15, 24, 33] are same as the results in [38]. In 2017, Zou [48] presented an unconditionally stable quadratic scheme. In 2020, Zhou and Wu [45, 46] improved some coercivity results, e.g., the minimum angle condition for the quadratic scheme presented in [36] is improved to $1.42°$. Therefore, most existing coercivity results of quadratic schemes required a certain minimum angle condition. Some studies about other types of hybrid FVMs and Hermite FVMs were presented in [6, 11, 12] and so on, and some coercivity results on quadrilateral meshes can be found in [7, 18, 22, 23, 29, 32, 41, 44, 47] and the references cited therein. Once the coercivity result is proved, the error estimate in H^1 norm is then routine.

Received by the editors January 18, 2020 and, in revised form, October 2, 2020.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.

However, the L^2 error estimate is much more difficult compared with H^1 error analysis. For example, for the linear FVEM over triangular meshes, by assuming that the exact solution $u \in W^{3,p}(\Omega)$, $p > 1$, in 1994 Chen [9] given a proof of optimal convergence order of L^2 error estimate. In 1998, Huang and Xi [19] present a counterexample to show that the optimal second order accuracy of L^2 error norm cannot be achieved by only assuming that $(u, f) \in H^2 \times L^2$, where f is the right hand side function. In 2002, Chen, Li and Zhou [10] discussed the optimal L^2 error estimate of linear FVEM, they proved that the L^2 error can be bounded by $h^2|\ln h|^{1/2}\|f\|_{1,1}$ and $h^2\|f\|_{1,p}$, $p > 1$. In 2002, Ewing, Lin and Lin [16] proved that the L^2 error can be bounded by $h^2\|u\|_2 + h^{1+\beta}\|f\|_\beta$ for $0 < \beta \leq 1$. Recently, by introducing two orthogonal conditions on triangular meshes, in 2016 Wang and Li [36] constructed k-order FVE schemes such that the L^2 error can be bounded by $h^{k+1}\|u\|_{k+2}$. Some L^2 error estimates on quadrilateral meshes can be found in [26, 27, 28, 29, 42] for an incomplete references.

Therefore, one can observe that the theoretical analysis of quadratic FVEM on triangular meshes has not been developed satisfactorily. On the one hand, under a certain minimum angle condition $1.42°$ for the isotropic diffusion problems, the quadratic scheme in [36] can ensure the optimal L^2 error order. On the other hand, the unconditionally stable quadratic scheme presented in [48] does not guarantee the optimal convergence order in L^2 norm, at least it seems so numerically. Namely, in the exiting literature the quadratic scheme in [36] is the unique scheme such that the optimal L^2 error estimate holds, while the scheme in [48] is the unique unconditionally stable scheme. At this stage, in order to satisfy the wide applications of quadratic FVEM (e.g. [8, 14, 20, 35, 37, 39, 40]), it requires us make efforts to construct an unconditionally stable quadratic FVE scheme with the optimal L^2 error estimate. However, by the existing analysis techniques, it is difficult to find a quadratic FVE scheme such that the local coercivity result (independent of the minimum angle of underlying mesh) and optimal L^2 error estimate holds simultaneously.

Unlike the existing quadratic FVE schemes, in this work we propose a class of bubble enriched quadratic FVE schemes such that the H^1 (resp. L^2) error order is 2 (resp. 3). Precisely, by adding a space which consists of element-wise bubble functions to the standard quadratic finite element space, we obtain a class of FVE schemes with three scheme parameters α, β_1 and β_2, where $\alpha \in (0, 1/2)$ on the element boundary and $0 < \beta_1 < 2/3 < \beta_2 < 1$ in the interior of element. For some special schemes, by element analysis, we numerically show that the local coercivity result is valid on a class of isosceles triangles. When $\alpha = (3 - \sqrt{3})/6$, β_1 and β_2 satisfy (15) (or equivalently (18)), under the coercivity result, we proved that $|u - u_h|_1 = \mathcal{O}(h^2)$. Moreover, by the Aubin-Nitsche technique and assuming that $u \in H^3(\Omega)$, $f \in H^2(\Omega)$, we proved that $\|u - u_h\|_0 = \mathcal{O}(h^3)$ for these schemes. Finally, the theoretical results are verified by several numerical examples.

For the class of schemes presented in this paper, we may find a scheme such that the local coercivity result holds in future. Moreover, the existence of a class of FVE schemes can be used to attack many complicated problems. For example, we may search this class for a scheme which satisfied some additional properties, such as the positivity-preserving.

The rest part of this paper is organized as follows. In Section 2 we present a class of bubble enriched quadratic FVE schemes on triangular meshes. The coercivity result and H^1 error estimates of these schemes are discussed in Section 3. In Section 4, we proved that the L^2 error order is 3 of these schemes when the scheme