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A FAST AND HIGH ACCURACY NUMERICAL SIMULATION FOR A FRACTIONAL BLACK-SCHOLES MODEL ON TWO ASSETS*†

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Abstract

In this paper, a two dimensional (2D) fractional Black-Scholes (FBS) model on two assets following independent geometric Lévy processes is solved numerically. A high order convergent implicit difference scheme is constructed and detailed numerical analysis is established. The fractional derivative is a quasi-differential operator, whose nonlocal nature yields a dense lower Hessenberg block coefficient matrix. In order to speed up calculation and save storage space, a fast bi-conjugate gradient stabilized (FBi-CGSTAB) method is proposed to solve the resultant linear system. Finally, one example with a known exact solution is provided to assess the effectiveness and efficiency of the presented fast numerical technique. The pricing of a European Call-on-Min option is showed in the other example, in which the influence of fractional derivative order and volatility on the 2D FBS model is revealed by comparing with the classical 2D B-S model.

Keywords 2D fractional Black-Scholes model; Lévy process; fractional derivative; numerical simulation; fast bi-conjugrate gradient stabilized method

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1 Introduction

During the last few decades the scientists have had a considerable interest in

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modelling financial markets and pricing financial derivatives. The pioneering work was presented in the 1970s by Black, Scholes [4] and Merton [23], who set up the key principles of no arbitrage option pricing and derived a well known differential equation model, that is the B-S model. However, the model was proposed under many strict assumptions on the real financial market. More general models by relaxing some restrictions were constructed in ongoing researches [12, 13, 24].

It is frequently found that the option price presents the features of heavy tails and volatility skew or smile in real markets. Gaussian models fail to describe these phenomena while Lévy (or α-stable) distributions can do. Lévy processes allow extreme but realistic events, such as sudden jumps of market prices [7, 19, 20]. Therefore, more and more different Lévy processes have been introduced into the financial field to model price of financial derivatives. A modified Lévy-α-stable process was proposed by Koponen [17] and Boyarchenko and Levendorskiǐ [5] to model the dynamics of securities. This modification yields a damping effect in the tails of the Lévy stable distribution, which was known as the KoBoL process. Carr, Geman, Madan and Yor [6] raised a Lévy process (that is the CGMY process), which allowed for jump components displaying both finite and infinite activity and variation. A finite moment log stable (FMLS) process with the tail index $\alpha \in (0, 2]$, which can capture the highly skewed feature of the implied density for log returns, was applied to model S&P 500 option prices by Carr and Wu [7]. Schoutens [27, 28] summed up the application of Lévy process in finance. Of all the Lévy processes, the most interesting include the CGMY, KoBoL and FMLS processes.

Fractional derivatives are quasi-differential operators, which provide useful tools for a description of memory and hereditary properties and are closely related to Lévy processes. When the price log-returns are driven by a Lévy fractional stable distribution, after some suitable transformations, the price of an option on underlying assets can be modeled by a fractional partial differential equation (FPDE) [1, 8–10, 14, 34]. In this paper we focus on a 2D FBS model governing European Call-on-Min option. Assuming the two underlying assets S_1 and S_2 follow two independent geometric Lévy processes with maximal negative asymmetry (or skewness) [7] (that is the FMLS process), the option price V on these two assets is determined by a 2D FBS equation [11] as below

$$\frac{\partial V}{\partial t} + (r - v_\alpha)\frac{\partial V}{\partial x} + (r - v_\beta)\frac{\partial V}{\partial y} + v_\alpha \cdot {}_{-\infty}D_x^\alpha V + v_\beta \cdot {}_{-\infty}D_y^\beta V = rV, \tag{1}$$

where $x = \ln S_1$, $y = \ln S_2$, $v_\alpha = -\frac{1}{2}\sigma^\alpha \sec\frac{\alpha\pi}{2}$, $v_\beta = -\frac{1}{2}\sigma^\beta \sec\frac{\beta\pi}{2}$, and the parameters r and $\sigma (\geq 0)$ are the risk-free rate and the volatility of the returns from the holding stock price, respectively. The fractional operators ${}_{-\infty}D_x^\alpha$ and ${}_{-\infty}D_y^\beta$ $(1 < \alpha, \beta \leq 2)$ are the left Riemann-Liouville fractional derivatives defined on infinite