Ann. of Appl. Math.
35:4(2019), 357-363

SEYMOUR'S SECOND NEIGHBORHOOD IN 3-FREE DIGRAPHS *

Bin Chen[†] An Chang

(*Center for Discrete Math. and Theoretical Computer Science, Fuzhou University, Fuzhou 350108, Fujian, PR China*)

Abstract

In this paper, we consider Seymour's Second Neighborhood Conjecture in 3-free digraphs, and prove that for any 3-free digraph D, there exists a vertex say v, such that $d^{++}(v) \geq \lfloor \lambda d^{+}(v) \rfloor$, $\lambda = 0.6958\cdots$. This slightly improves the known results in 3-free digraphs with large minimum out-degree.

Keywords Seymour's second neighborhood conjecture; 3-free digraph

2000 Mathematics Subject Classification 15A42; 05C50

1 Introduction

All digraphs considered in this paper are finite, simple and digonless. Let $D = (V, A)$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any vertex $v \in V$, the out-neighbourhood of v is the set $N^{+}(v)$=$\{u \in V(D)$:$(v, u) \in A(D)\}$, and the out-degree of v is $d^{+}(v)$= $|N^{+}(v)|$. The in-neighbourhood of v is the set $N^{-}(v)$=$\{u \in V(D)$:$(u, v) \in A(D)\}$, and the in-degree of v is $d^{-}(v)$= $|N^{-}(v)|$. The set $N(v) = N^{+}(v) \cup N^{-}(v)$ is called the neighbourhood of v. We call the vertices in $N^{+}(v), N^{-}(v)$ and $N(v)$ the out-neighbours, in-neighbours and neighbours of v respectively. The minimum out-degree (minimum in-degree) of D is $\delta^{+}(D) = \min\{d^{+}(v) : v \in V(D)\}$ ($\delta^{-}(D) = \min\{d^{-}(v) : v \in V(D)\}$). For a set $S \subseteq V$, we let $N^{+}(S) = \bigcup\limits_{v \in S} N^{+}(v) - S$, $N^{-}(S) = \bigcup\limits_{v \in S} N^{-}(v) - S$. For any vertex v, let $N^{++}(v) = N^{+}(N^{+}(v))$ and $d^{++}(v) = |N^{++}(v)|$. Similarly, one can define the maximum out-degree of D, $\Delta^{+}(D)$, and the maximum in-degree, $\Delta^{-}(D)$, $N^{--}(v)$, $d^{--}(v)$.

For the purpose of this paper, all cycles considered here are direct cycles. The girth $g(D)$ of D is the minimum length of the cycles of D. A digraph D is k-free means that $g(D) \geq k + 1$ for $k \geq 2$, that is, there is no cycle whose length is less than k in D.

In 1990, Seymour [1] put forward the following conjecture:

---

*Manuscript received September 17, 2019

[†]Corresponding author. E-mail: cbfzu03@163.com

Conjecture 1.1(Seymour's Second Neighborhood Conjecture) *For any digraph D, there exists a vertex v such that $d^{++}(v) \geq d^{+}(v)$.*

We call the vertex v in Conjecture 1.1 a Seymour vertex. In 1996, Fisher [3] proved that Conjecture 1.1 is true if D is a tournament. In 2007, Fidler and Yuster [2] showed that any tournament minus a star or a sub-tournament, and any digraph D with minimum degree $|V(D)|-2$ has a Seymour vertex. In 2016, Cohn et al [8] proved that almost surely there are a large number of Seymour vertices in random tournaments and even more in general random digraphs. However, Conjecture 1.1 is still an open problem for general digraphs.

Another approach to Conjecture 1.1 is to determine the maximum value of λ such that there is a vertex v in D satisfying $d^{++}(v) \geq \lambda d^{+}(v)$ for any digraph D. Chen, Shen and Yuster [4] proved that $d^{++}(v) \geq \lambda d^{+}(v)$, $\lambda = 0.6572\cdots$ is the unique real root of the equation $2x^3+x^2-1=0$. In 2010, Zhang and Zhou [7] proved that for any 3-free digraph D, there exists a vertex v in D such that $d^{++}(v) \geq \lambda d^{+}(v)$, where $\lambda = 0.6751\cdots$ is the only real root in the interval $(0,1)$ of the polynomial $x^3+3x^2-x-1=0$. Liang and Xu [6] considered k-free digraphs, $k \geq 3$, and proved that $d^{++}(v) \geq \lambda_k d^{+}(v)$, where λ_k is the only real root in the interval (0,1) of the polynomial

$$g_k(x) = 2x^3-(k-3)x^2+(2k-4)x-(k-1).$$

Furthermore, λ_k is increasing with k, and $\lambda_k \to 1$ while $k \to \infty$. When k=3, $\lambda_3 = 0.6823\cdots$ is the only real root in the interval $(0,1)$ of the polynomial $x^3+x-1=0$.

In this paper, we consider Seymour's second neighborhood conjecture in 3-free digraphs, and our result slightly improves the known results in 3-free digraphs with large minimum out-degree.

Theorem 1.1 *Let D be an n order 3-free digraph, then there exists a vertex $v \in V(D)$ such that $d^{++}(v) \geq \lfloor \lambda d^{+}(v) \rfloor$, where $\lambda = 0.6958\cdots$ is the only real root in the interval $(0,1)$ of the polynomial $x^3+\frac{1}{2}x^2-\frac{(1-x)^2}{1.17}-\frac{1}{2}=0$.*

This paper is organized as follows. In Section 2, we first introduce some definitions and notations used in the paper, and give some lemmas in order to prove Theorem 1.1. In Section 3, we will prove Theorem 1.1.

2 Preparation

A digraph G is a subdigraph of a digraph D if $V(G) \subseteq V(D)$, $A(G) \subseteq A(D)$. For any subdigraph G of D, let $N_G^+(v) = N_D^+(v) \cap V(G)$ and $d_G^+(v) = |N_G^+(v)|$. For a set $W \subseteq V$, we let $D[W]$ denote the subgraph induced by W and $N_W^+(v) = N_{D[W]}^+(v)$, $d_W^+(v) = d_{D[W]}^+(v)$. Similarly, we can define $N_G^-(v)$, $d_G^-(v)$, $N_W^-(v)$, $d_W^-(v)$. For any two vertex disjoint vertex sets X and Y, denote $A(X,Y)$ as the arc set between X and Y, every arc $(x,y) \in A(X,Y)$ with $x \in X$ and $y \in Y$.