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A Fast Algorithm for Time-Dependent Radiative Transport Equation Based on Integral Formulation

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Abstract. In this work, we introduce a fast numerical algorithm to solve the time-dependent radiative transport equation (RTE). Our method uses the integral formulation of RTE and applies the treecode algorithm to reduce the computational complexity from $\mathcal{O}(M^{2+1/d})$ to $\mathcal{O}(M^{1+1/d}\log M)$, where M is the number of points in the physical domain. The error analysis is presented and numerical experiments are performed to validate our algorithm.

AMS subject classifications: 45K05, 65N22, 65N99, 65R20, 65Y10

Key words: Radiative transport equation, volume integral equation, treecode algorithm.

1 Introduction

The radiative transport model plays an important role in quantitative modeling and analysis of particle transport processes in many physical and biological applications such as astrophysics [5, 19], nuclear engineering [23, 30], biomedical optics [1, 2, 25, 33, 41], radiation therapy [20, 37]. In this paper, we consider the numerical solution to the time-dependent radiative transport equation (RTE) with isotropic scattering kernel:

$$\begin{aligned} u_t(t,\mathbf{x},\mathbf{v})+[\mathbf{v}\cdot\nabla+\sigma_t(\mathbf{x})]\,u(t,\mathbf{x},\mathbf{v}) &= \sigma_s(\mathbf{x})\langle u\rangle(t,\mathbf{x})+f(t,\mathbf{x}) && \text{in } (0,T]\times\Omega\times\mathbb{S}^{d-1}, \\ u(t,\mathbf{x},\mathbf{v}) &= 0 && \text{on } \{0\}\times\Omega\times\mathbb{S}^{d-1}, \\ u(t,\mathbf{x},\mathbf{v}) &= 0 && \text{on } (0,T]\times\Gamma_-, \end{aligned} \tag{1.1}$$

where the space $\Omega\subset\mathbb{R}^d$ is a convex domain with smooth boundary $\partial\Omega$, $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$. $\Gamma_-:=\{(\mathbf{x},\mathbf{v})\in\partial\Omega\times\mathbb{S}^{d-1}\,|\,\mathbf{v}\cdot\mathbf{n}_\mathbf{x}<0\}$ ($\mathbf{n}_\mathbf{x}$ being the unit outward normal at $\mathbf{x}\in\partial\Omega$) is the incoming boundary set. $\sigma_t(\mathbf{x})$ and $\sigma_s(\mathbf{x})$ are the total absorption and scattering coefficients, respectively. Physically speaking, the coefficient $\sigma_s(\mathbf{x})$ represents

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the strength of the scattering of the underlying medium at $\mathbf{x}\in\Omega$ and $\sigma_a(\mathbf{x}):=\sigma_t(\mathbf{x})-\sigma_s(\mathbf{x})$ represents the strength of absorption of the medium. $f(t,\mathbf{x})$ is a time-dependent isotropic source function (which is not dependent on $\mathbf{v}$). The quantity $\langle u\rangle(t,\mathbf{x})$ is defined by

$$\langle u\rangle(t,x):=\int_{\mathbb{S}^{d-1}} u(t,\mathbf{x},\mathbf{v}')d\mathbf{v}', \tag{1.2}$$

where $d\mathbf{v}'$ is the *normalized* surface measure on $\mathbb{S}^{d-1}$. For the sake of simplicity, we have assumed there is no incoming source on the boundary, and the solution $u(t,\mathbf{x},\mathbf{v})$ is zero at $t=0$.

The analytic solutions for the time-dependent RTE (1.1) have only been found in special setup, such as for homogeneous infinite or semi-infinite geometries [10,26,31], and layered media [27]. Numerical methods for solving (1.1) has been extensively explored, see [16,17,22,24,36] and references therein for an overview. These numerical algorithms are mainly based on stochastic Monte Carlo [15,17,29], discrete ordinate [6,13,18,21,28], or P_N formulation [9,32]. The most challenging issue for solving the RTE numerically is due to the high dimensionality of the phase space that includes both physical and angular dimensions. Regarding time-independent problems, one of the popular ways is based on the integral formulation to remove the angular variable by computing the angular moments [11,34,35]. For isotropic scattering media, the fast algorithms based on fast multipole method [34] and low rank matrix factorization [11] were developed. For anisotropic scattering media, a truncated coupled system of integral equations for the angular moments of the transport solution were studied in [35]. Particularly, for those highly separable scattering phase functions such as Rayleigh or linearly anisotropic cases, the integral formulation could solve the RTE very effectively by exploiting the low rank structure of integral kernels [35]. Regarding time-dependent problems as (1.1), the integral formulation for infinite homogeneous medium has been carried out in [36,40], however the related fast algorithms have not been addressed yet.

In our work, we will pursue the integral formulation for angular averaged solution for time-dependent RTE and develop a fast solver based on the treecode algorithm for the resulting integral equation in space and time, which is more complicated due to the manifold structure, a conical surface, for the domain of dependence. We will briefly derive the integral formulation in Section 2 and provide a few mathematical preliminaries in Section 3. Then we present our fast algorithm including discretization, error analysis, and implementation details in Section 4. We provide numerical experiments in Section 5 and concluding remarks in Section 6.

2 Integral formulation

In this section, we first briefly introduce the integral formulation for the time-dependent RTE (1.1). Let

$$\mathbf{z}:=(t,\mathbf{x})\in\mathbb{R}^{d+1},\quad \boldsymbol{\theta}:=(1,\mathbf{v})\in\{1\}\times\mathbb{S}^{d-1}. \tag{2.1}$$