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Dynamics of a Non-Linear Stochastic Viscoelastic Equation with Multiplicative Noise

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Abstract. The well-posedness and stability properties of a stochastic viscoelastic equation with multiplicative noise, Lipschitz and locally Lipschitz nonlinear terms are investigated. The method of Lyapunov functions is used to investigate the asymptotic dynamics when zero is not a solution of the equation by using an appropriate cocycle and random dynamical system. The stability of mild solutions is proved in both cases of Lipschitz and locally Lipschitz nonlinear terms. Furthermore, we investigate the existence of a non-trivial stationary solution which is exponentially stable, by using a general random fixed point theorem for general cocycles. In this case, the stationary solution is generated by the composition of random variable and Wiener shift. In addition, the theory of random dynamical system is used to construct another cocycle and prove the existence of a random fixed point exponentially attracting every path.

Key Words: Stochastic viscoelastic; exponential stability; stabilization; random dynamical systems; attractors.

1 Introduction

This paper focuses on the following stochastic viscoelastic partial differential equation

$$\begin{cases} u_{tt}-u_{xx}-u_{xxt}=f(u)+g(u)\xi, & (x,t)\in(0,l)\times(0,\infty),\\ u(x,0)=h_0(x), \quad u_t(x,0)=h_1(x), & x\in(0,l),\\ u(0,t)=u(l,t)=0, & t>0, \end{cases} \tag{1.1}$$

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where ξ is a space-time white-noise, in particular, we will consider $\xi = \dot{W}$, where W is a Q-Wiener process (see [1]), f and g are Lipschitz continuous functions with $L_f \geq 0$ and $L_g \geq 0$ their respective Lipschitz constants. This problem arises in mechanics with a particular damping for nonlinear viscoelastic system where we suppose that the stress is of rate type, i.e., $\sigma = u_x + u_{xt}$. Then, the momentum equation is written as (1.1) and models the longitudinal vibrations of a bar subjected to viscous effects. The term $-u_{xxt}$ indicates that the stress is proportional to the strain rate as in a linearized Kelvin-Voigt material. One motivation to study exponential stability comes from the control theory, for instance controlling the vibration of a bridge when is driven by a force that depends on a strain which possesses a random term. Furthermore, this problem, in its deterministic version was considered as a hyperbolic problem, however the study of its properties (such as regularity or stability) is identified as a parabolic problem (see [2] for more details). The existence, uniqueness and regularity of solutions have been analyzed in the field of stochastic partial differential equations, from the point of view of the semigroups theory, for example in [1].

Our emphasis here will be paid to study the asymptotic behavior. To this end we will exploit two methods. The first one is based on the Lyapunov techniques and the properties of the semigroup associated with our problem, a general reference here is [3] and the references therein. The second method uses cocycles and the theory of random dynamical systems (RDS) (see [4] for details) in order to analyze the stability of mild solutions when f and g are locally Lipschitz nonlinear terms.

A large amount of studies have been carried out towards the dynamics of a variety of systems related to strongly damped wave equations. For example, the asymptotical behavior of solutions for deterministic and stochastic strongly damped wave equations has been studied by many authors under different types of hypotheses, see, e.g. [5–13] and references therein.

But, to the best of our knowledge, there were no results in the case of a stochastic strongly damped wave equations with $f(0) \neq 0$, without dynamical friction u_t, and under locally Lipschitz assumption on the operators f and g. In our current paper we analyze this case and we first prove exponential stability of the null solution by using the Lyapunov technique, when $f(0) = g(0) = 0$ and under locally Lipschitz assumptions. Second, in the case of $f(0) \neq 0$, we use the techniques of random dynamical systems to prove the existence of a stationary solution which is exponentially attracting.

The paper is organized as follows. In Section 2 we introduce some results concerning the stability of deterministic and stochastic evolution equations. Section 3 is first devoted to the exponential stability analysis by means of the Lyapunov method in the cases of global and locally Lipschitz nonlinear terms; then, thanks to the theory of cocycles and RDS, we construct a cocycle associated to our problem and prove the existence of a solution which is mean square (and also almost surely) exponentially attracting, then we construct a conjugated cocycle which possesses a singleton random attractor which exponentially attracts every path. However we need to consider a more specific noisy term