J. Math. Study
doi: 10.4208/jms.v52n4.19.02

Vol. **52**, No. 4, pp. 378-393
December 2019

Computational Zooming in Near Unilateral Cracks by Schwarz Method with Total Overlap

Faker Ben Belgacem[1], Nabil Gmati[2] and Faten Jelassi[1,*]

[1] *Alliance Sorbonne Université , UTC, EA 2222, Laboratoire de Mathématiques Appliquées de Compiègne, F-60205 Compiègne, France.*

[2] *College of Sciences & Basic and Applied Scientific Research Center, Imam Abdulrahman Bin Faïsal University, Dammam, Saudi Arabia.*

Received January 30, 2019; Accepted April 27, 2019

Abstract. We focus on the numerical solver of unilateral cracks by the Schwarz Method with Total Overlap. The aim is to isolate the treatment at the vicinity of the cracks from other regions of the computational domain. This avoids any direct interaction between specific approximations one may use around the singularities born at the tips of the cracks and more standard methods employed away from the cracks. We apply an iterative sub-structuring technique to capture the small structures by insulating the cracks into patches and making a zoom around each of them. The macro-problem is in turn set on the whole domain. As for the classical Schwarz method, the communication between the micro (local) and macro (global) levels is achieved iteratively through some suitable boundary conditions. The micro problem is fed by Dirichlet data along the (outer) boundary of the patches. The specificity of our approach is that the macro problem inherits transmission conditions. Although they are expressed across the cracks, the final algebraic system to invert is blind to the discontinuities of the solution. In fact, the stiffness matrix turns out to be the one related to a safe domain, as if cracks were closed or the unilateral singularities were switched off. Only the right hand side is affected by what happens at the vicinity of the cracks. This enables users to run one of many efficient algorithms found in the literature to solve the linear macro-problem. In the other hand side, in spite of the still bad conditioning and the non-linearity of the unilateral micro problems, they are reduced in size and may be inverted properly by convex optimization algorithms. A successful convergence analysis of this variant of the Schwarz Method is performed after adapting to the unilateral non-linearity the variational tools developed by P. L. Lions.

AMS subject classifications: 35N86, 65N55

---

*Corresponding author. *Email addresses:* `faker.ben-belgacem@utc.fr` (F. Ben Belgacem), `faten.jelassi@utc.f` (F. Jelassi), `nmgmati@iau.edu.sa` (N. Gmati)

Key words: Integral operators, regular kernels, Jacobi transform, separated variables approximation.

1 Introduction

Realistic cracks that occur in many engineering fields are most often subjected to non linear unilateral contact conditions. Designing affordable methodologies for their reliable numerical simulation still arises substantial interest. Multifold enhancements may be on increasing the accuracy of the approximation, on improving the efficiency of the solvers to run for evaluating the discrete solution or on the implementation and parallelization procedures of these solvers, ... etc.

Enrichments of finite element methods have been elaborated to strengthen the accuracy of the discrete displacement, strain and stress fields. A qualitative analysis of the singularities arisen at the vicinity of crack tips is now known and practitioners can therefore refine the mesh in a well adapted shape so to well capture the singular part of the solution. Newer procedures may also be employed to improve the computed solution (see [5,13,14,32,33,35,37]). We point out, in particular, different versions of X-fem, PU-fem and G-Fem with or without (Heaviside) cut-off functions (see [2,9,10,36]). The reverse of the medal has to do with the condition number of the discrete problems to solve. These problems suffer from bad and possibly very bad conditioning. We refer for instance to [2,10] for some convincing examples. Solving efficiently the discrete system, after carrying out a finite element approximation of any kind, requires some known effective techniques such as multi-grid, multi-scale or sub-structuring approaches. This is the way followed for instance in [7,15,24,26,38,39]. We pursue here a similar objective by applying the iterative Schwarz Method with Total Overlap for the unilateral cracks introduced in for linear problems in [6,23]. The aim here is the realization, at each iteration, of a numerical zoom where it is needed, in a thin region surrounding the cracks. Zooming methods has been used in the finite elements analysis of the linear case and proved to be helpful in well capturing the localized singularities. We recommend [3,4,7,8,11,15,16,20,22,28,29,31,34] and references therein.

The purpose is hence to apply the Schwarz Method with Total Overlap to compute the displacement or temperature field at the vicinity of cracks where the contact is of Signorini's type. It is successfully used and analyzed for linear cracks in [6]. It helps making a numerical zoom around the geometrical singularities, born along and especially at the tips of the cracks. The displacement field satisfies then a variational inequality within a patch surrounding the cracks. To complete the construction of the full solution, a linear problem is solved in the whole domain. Both problems talk to each other alternately through some suitable transmission conditions. How the coupling operates is the distinctive marker of the method. The core advantage is related to the linear problem which can be set up on the safe domain. Things occur as if the cracks are closed, at least for the stiffness matrix. The straightforward result is the enhancing of the condition number of