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Structural Stability of $p(x)$-Laplace Problems with Fourier Type Boundary Condition

KANSIE Kpe[1] and OUARO Stanislas[2,*]

[1] *LAboratoire de Mathématiques et Informatique (LA.M.I), UFR. Sciences et Techniques, Université Nazi Boni, 01 BP 1091 Bobo 01, Bobo-Dioulasso, Burkina Faso.*
[2] *LAboratoire de Mathématiques et Informatique (LA.M.I), UFR. Sciences Exactes et Appliquées, Université Joseph KI ZERBO, 03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso.*

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Abstract. We study the continuous dependence on coefficients of solutions of the nonlinear nonhomogeneous Fourier boundary value problems involving the $p(x)$-Laplace operator.

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1 Introduction

In this paper, we study the convergence of sequences of solutions of degenerate elliptic problems with variable coercivity and growth exponents p_n of the form

$$(Pb_n):\begin{cases} b(u_n)-\mathrm{div}a_n(x,\nabla u_n)=f_n & \text{in } \Omega,\\ a_n(x,\nabla u_n)\cdot\eta+\lambda_n u_n=g_n & \text{on } \partial\Omega,\end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ ($N\geq 3$) with smooth boundary $\partial\Omega$ and η is the outer unit normal to $\partial\Omega$. Here, $b:\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous, onto and non-decreasing

*Corresponding author. *Email addresses:* `kansiek@yahoo.fr` (K. Kansie), `ouaro@yahoo.fr` (S. Ouaro)

function such that $b(0)=0$; $(a_n(x,\xi))_{n\in\mathbb{N}}$ is a family of applications which verify the classical Leray-Lions hypotheses but with a variable summability exponent $p_n(x)$ converging in measure to some exponent p such that $1<p_-\le p_n(\cdot)\le p_+<+\infty$, $(f_n)_{n\in\mathbb{N}}\subset L^1(\Omega)$, $(g_n)_{n\in\mathbb{N}}\subset L^1(\partial\Omega)$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers. The model problem for our study is so the following:

$$(Pb):\begin{cases} b(u)-\mathrm{div}a(x,\nabla u)=f & \text{in } \Omega,\\ a(x,\nabla u)\cdot\eta+\lambda u=g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\ge 3)$ is an open bounded domain with smooth boundary $\partial\Omega$ and η is the outer unit normal to $\partial\Omega$.

This paper is inspired by recent works of Andreianov, Bendahmane and Ouaro (see [1]) on the structural stability of weak and renormalized solutions u_n of the following nonlinear homogeneous Dirichlet boundary value problem

$$\begin{cases} b(u_n)-\mathrm{div}a_n(x,\nabla u_n)=f_n & \text{in } \Omega,\\ u_n=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $(a_n(x,\xi))_{n\in\mathbb{N}}$ verifies the classical Leray-Lions hypotheses with the variable exponents $p_n(x)$ such that $1<p_-\le p_n(\cdot)\le p_+<+\infty$. In their investigations, the exponent p_n (and thus, the underlying function space for the solution u_n) varying with n, the convergence of weak solutions u_n requires some involved assumptions on the convergence of the sequence f_n of the source terms. To bypass this difficulty, they used the technique of renormalized solutions. Indeed, the study of convergence of renormalized solutions of the problem (1.2) permits them to deduce convergence results for the weak solutions under much simpler assumptions on $(f_n)_{n\in\mathbb{N}}$, in particular the weak L^1 convergence of f_n to a limit f sufficiently regular so that to allow for the existence of a weak solution. Moreover, the structural stability results permits them to deduce also new existence results of solutions.

Problems with variable exponents $p(x)$ and $p_n(x)$ were arisen and studied by Zhikov in the pioneering paper [2]. The study of problems involving variable exponent has received considerable attention in recent years due to the fact that they can model various phenomena which arise in the study of elastic mechanics, electrorheological and thermorheological fluids (see [3–6]) or image restauration (see [7,8]).

Let us give the outline of the paper. In Section 2, we introduce some preliminary results. In Section 3, we prove the existence and uniqueness of the renormalized solution of (1.1) with L^1-data f and g. In Section 4, we tackle the question of continuous dependence for renormalized solutions.