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A RELAXATION APPROACH TO DISCRETIZATION OF BOUNDARY OPTIMAL CONTROL PROBLEMS OF SEMILINEAR PARABOLIC EQUATIONS

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Abstract. We consider an optimal boundary control problem described by a semilinear parabolic partial differential equation, with control and state constraints. Since this problem may have no classical solutions, it is reformulated in the relaxed form. The relaxed control problem is discretized by using a finite element method in space and a partially implicit scheme in time, while the controls are approximated by piecewise constant relaxed controls. We first state the necessary conditions for optimality for the continuous problem and the discrete relaxed problem. Next, under appropriate assumptions, we prove that accumulation points of sequences of optimal (resp. admissible and extremal) discrete relaxed controls are optimal (resp. admissible and extremal) for the continuous relaxed problem.

Key words. Boundary optimal control, semilinear parabolic systems, state constraints, relaxed controls, discretization.

1. Introduction

It is well known that optimal control problems, without any convexity assumptions on the data, have no classical solutions in general. These problems are usually studied by considering their corresponding relaxed formulations, where at each time, the control variable is not a vector in some set but instead a probability measure on that set. Relaxation theory has been introduced, initially, in order to prove existence of optimal controls and later to derive necessary conditions for optimality. There exist an extensive literature concerning relaxation of control problems, see e.g. Warga [19], Roubíček [16], Fattorini [11] and the references therein.

In this paper we consider an optimal boundary control problem for systems governed by a semilinear parabolic partial differential equation, with control and state constraints. The problem is motivated, for example, by the control of a heat (or other) diffusion process whose source is nonlinear in the heat and temperature, with nonconvex cost and control constraint set (e.g. on-off type control). This class of problems has been extensively studied by several authors, among them Ahmed et al. [1], Casas [5], Barbu [2], Fattorini et al. [10], Tröltzsch [18] etc. We first state the existence of optimal controls and the necessary conditions for optimality for the continuous relaxed problem. Then, the relaxed problem is discretized by using a Galerkin finite element method with continuous piecewise linear basis functions in space for space approximation, and a partially implicit scheme in time, while the controls are approximated by piecewise constant relaxed controls. The discretization is motivated by the fact that in practice optimization methods are usually applied to the problem after some discretization. Then, we prove the existence of optimal controls and derive necessary conditions for optimality for the discrete relaxed problem. Finally, we study the behaviour in the limit of the above approximation. More precisely, we prove, under appropriate assumptions, that accumulation points of sequences of optimal (resp. admissible and extremal) discrete

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relaxed controls are optimal (resp. admissible and extremal) for the continuous relaxed problem. The novelty of the present paper is in the finite element approximation of a boundary optimal control problem using, as a tool, relaxed controls, which can be further used in optimization algorithms (see [8]). For a different approach, using differential inclusions and approximations in abstract spaces, of a Mayer type optimal control problem, see Mordukhovich et al. [14], where existence theory, necessary optimality conditions and convergence are considered.

For approximation of nonconvex optimal control and variational problems, and of Young measures, see e.g. [4, 6, 9, 13, 15] and the references therein.

2. The continuous optimal control problems

Let Ω be a bounded domain in $\mathbb{R}^d$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, $I = (0,T)$, $T < \infty$, an interval, and set $Q := \Omega \times I$, $\Sigma_0 := \Gamma_0 \times I$, $\Sigma_1 := \Gamma_1 \times I$ and $\Sigma := \Gamma \times I$. Consider the parabolic state equation

$$y_t + A(t)y = f_0(x,t,y(x,t)) \text{ in } Q, \tag{1}$$

$$y(x,t) = 0 \text{ on } \Sigma_0, \tag{2}$$

$$\frac{\partial y}{\partial \nu_A} = f_1(x,t,w(x,t)) \text{ on } \Sigma_1, \tag{3}$$

$$y(x,0) = y^0(x) \text{ in } \Omega, \tag{4}$$

where $A(t)$ is the second order elliptic differential operator

$$A(t)y := -\sum_{j=1}^{d}\sum_{i=1}^{d} (\partial/\partial x_i)[a_{ij}(x,t)\partial y/\partial x_j] \tag{5}$$

and

$$\frac{\partial y}{\partial \nu_A} = \sum_{j=1}^{d}\sum_{i=1}^{d} a_{ij}(x,t)\frac{\partial y}{\partial x_j}\nu_j, \quad \text{with } (x,t) \in \Sigma_1, \tag{6}$$

where $\nu(x)$ is the outwards unit vector to Γ at the point x.

We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm in $L^2(\Omega)$, by $(\cdot,\cdot)_{\Gamma_1}$ and $\|\cdot\|_{\Gamma_1}$ the inner product and norm in $L^2(\Gamma_1)$, by $(\cdot,\cdot)_1$ and $\|\cdot\|_1$ the inner product and norm in the Sobolev space $H^1(\Omega)$ and by $<\cdot,\cdot>$ the duality bracket between $V := \{v \in H^1(\Omega) : v|_{\Gamma_0} = 0\}$, where $v|_{\Gamma_0}$ is the trace function on Γ_0 and its dual space V^*. The state equation will be interpreted in the following weak form

$$\begin{aligned} &< y_t, v > + a(t,y,v) = (f_0(t,y),v) + (f_1(t,w),v)_{\Gamma_1}, \ \forall v \in V, \text{ a.e. in } I, \\ &y(t) \in V \text{ a.e. in } I, y(0) = y^0, \end{aligned} \tag{7}$$

where the derivative y_t is understood in the sense of V-vector valued distributions, and $a(t,\cdot,\cdot)$ denotes the usual bilinear form on $V \times V$ associated with $A(t)$

$$a(t,y,v) := \sum_{j=1}^{d}\sum_{i=1}^{d}\int_\Omega a_{ij}(x,t)\frac{\partial y}{\partial x_i}\frac{\partial v}{\partial x_j}dx. \tag{8}$$

We define the set of *classical controls*

$$W := \{w : \Sigma_1 \to U \,|\, w \text{ measurable}\} \subset L^\infty(\Sigma_1),$$

where U is a compact subset of $\mathbb{R}^{d'}$, and the functionals

$$G_m(w) := \int_Q g_{0m}(x,t,y)dxdt + \int_{\Sigma_1} g_{1m}(x,t,y,w)d\gamma dt, \quad m = 0,...,q. \tag{9}$$