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The Fast Implementation of the ADI-CN Method for a Class of Two-Dimensional Riesz Space-Fractional Diffusion Equations

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Abstract. In this paper, a class of two-dimensional Riesz space-fractional diffusion equations (2D-RSFDE) with homogeneous Dirichlet boundary conditions is considered. In order to reduce the computational complexity, the alternating direction implicit Crank-Nicholson (ADI-CN) method is applied to reduce the two-dimensional problem into a series of independent one-dimensional problems. Based on the fact that the coefficient matrices of these one-dimensional problems are all real symmetric positive definite Toeplitz matrices, a fast method is developed for the implementation of the ADI-CN method. It is proved that the ADI-CN method is uniquely solvable, unconditionally stable and convergent with order $\mathcal{O}(\tau^2+h_x^2+h_y^2)$ in the discrete L_∞-norm with time step τ and mesh size h_x, h_y in the x direction and the y direction, respectively. Finally, several numerical results are provided to verify the theoretical results and the efficiency of the fast method.

AMS subject classifications: 35R11, 65M06, 65M12

Key words: Space-fractional diffusion equation, Riesz fractional derivative, alternating direction method, convergence and stability, L_∞-norm.

1 Introduction

The classical diffusion processes can be described properly by the Fick's law [1,2]. However, there are also a large number of diffusion processes found to be non-Fickian, which occur in many different areas, such as the signaling of biological cells [7], the design of photocopiers and laser printers [6], foraging behavior of animals [9], anomalous electro-diffusion in nerve cells [8], and electrochemistry [10], finance [12], physics [11], viscoelastic and viscoplastic flow [14], fluid and continuum mechanics [13] and solute transport

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in groundwater [15], etc. Many researchers show that fractional diffusion equations are more proper than second-order diffusion equations for the description of the anomalous diffusion processes [11,15,16]. As a result, many studies on fractional diffusion equation have emerged recently, such as [3,24–26] etc.

In this paper, we investigate the following 2D-RSFDE:

$$\frac{\partial u}{\partial t}-k_x\frac{\partial^\alpha u}{\partial|x|^\alpha}-k_y\frac{\partial^\beta u}{\partial|y|^\beta}=f(u,x,y,t),\quad (x,y)\in\Omega,\quad t\in(0,T], \tag{1.1}$$

subject to the initial condition

$$u(x,y,0)=\varphi(x,y),\quad (x,y)\in\Omega, \tag{1.2}$$

and the Dirichlet boundary condition

$$u(x,y,t)=0,\quad (x,y)\in\partial\Omega, \tag{1.3}$$

where $1<\alpha,\beta\leq 2$, $\Omega=(0,x_r)\times(0,y_r)$, $\partial\Omega$ is the boundary of Ω and $k_x>0$, $k_y>0$ are diffusion coefficients. In this paper the reaction term $f(u,x,y,t)$ is confined as $f(u,x,y,t)=g(x,y,t)-\mu u$ with $\mu>0$.

The space Riesz fractional derivative $\frac{\partial^\alpha u}{\partial|x|^\alpha}$ on the finite domain $\bar{\Omega}=\Omega\cup\partial\Omega$ is defined by

$$\frac{\partial^\alpha u}{\partial|x|^\alpha}(x,y,t)=-c_\alpha\left({}_0D_x^\alpha u(x,y,t)+{}_xD_{x_r}^\alpha u(x,y,t)\right),$$

where $c_\alpha=\frac{1}{2\cos(\pi\alpha/2)}$, and the left and right Riemann-Liouville fractional derivatives are defined as

$$\begin{aligned}
{}_0D_x^\alpha u(x,y,t)&=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_0^x\frac{u(s,y,t)ds}{(x-s)^{\alpha-1}},\\
{}_xD_{x_r}^\alpha u(x,y,t)&=\frac{(-1)^2}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^{x_r}\frac{u(s,y,t)ds}{(s-x)^{\alpha-1}}.
\end{aligned}$$

The space Riesz fractional derivative $\frac{\partial^\beta u}{\partial|y|^\beta}$ on the finite domain $\bar{\Omega}$ can be defined similarly.

Because of the nonlocal nature of the space fractional operators, numerical methods discretization of the Riesz space-fractional model always generate linear systems with full or dense coefficient matrices. Consequently, it will require an enormous amount of computer time for the numerical stimulations. Unfortunately, the amount of computer time increases even more for the two-dimensional case. In this paper, we apply the ADI method to reduce the multidimensional problem into a series of independent one-dimensional problems. Based on the fact that the coefficient matrices for these one-dimensional problems are all real symmetric positive definite matrices with the structure