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# Time-Stepping Error Bound for a Stochastic Parabolic Volterra Equation Disturbed by Fractional Brownian Motions

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**Abstract.** In this paper, we consider a stochastic parabolic Volterra equation driven by the infinite dimensional fractional Brownian motion with Hurst parameter $H \in [\frac{1}{2}, 1)$. We apply the piecewise constant, discontinuous Galerkin method to discretize this equation in the temporal direction. Based on the explicit form of the scalar resolvent function and the refined estimates for the Mittag-Leffler function, we derive sharp mean-square regularity results for the mild solution. The sharp regularity results enable us to obtain the optimal error bound of the time discretization. These theoretical findings are finally accompanied by several numerical examples.

**AMS subject classifications**: 60H15, 65M06
**Key words**: Stochastic parabolic Volterra equation, fractional Brownian motion, optimal error bound.

## 1. Introduction

In the last decades, numerical approximations of parabolic and hyperbolic stochastic partial differential equations (SPDEs) have been extensively studied (see, e.g. monographs [2, 3, 5, 6, 8, 12, 28–30, 34] and references therein). In contrast to the parabolic and hyperbolic SPDEs, stochastic Volterra equations (or fractional SPDEs) are much less well-understood, from both theoretical and numerical points of view. In the present work, we consider a stochastic parabolic Volterra equation (SPVE) driven by infinite dimensional fractional noise with Hurst parameter $H \in [\frac{1}{2}, 1)$, described by

$$u(t) + g_\alpha * Au(t) = W_Q^H(t) + u_0, \quad t \in (0, T], \tag{1.1}$$

where

$$(a * b)(t) := \int_0^t a(s) b(t-s)\, \mathrm{d}s$$

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denotes the convolution of $a$ and $b$ and $g_\alpha(t)$ serves as the fractional integral kernel $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$. The linear operator $A$ is assumed to be self-adjoint, positive and densely defined in a Hilbert space $\mathscr{H}$. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and let $\{W_Q^H(t)\}$ be an infinite dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in [\frac{1}{2},1)$, as will be defined in Section 2. The equation (1.1) can be regarded as the integral form of the following fractional stochastic parabolic equation

$$\begin{aligned} &\mathrm{d}u(t)+AD_t^{1-\alpha}u(t)\,\mathrm{d}t=\mathrm{d}W_Q^H(t) \quad \text{for } t\in(0,\ T] \\ &\quad \text{with} \quad u(0,\cdot)=u_0(\cdot) \ \text{ and } \ 0<\alpha<1. \end{aligned} \tag{1.2}$$

Here, $D_t^{1-\alpha}$ denotes the Riemann-Liouville fractional derivates with respect to $t$, defined by

$$D_t^{1-\alpha}u(t)=\frac{\partial}{\partial t}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)\,\mathrm{d}s, \quad \alpha\in(0,1) \tag{1.3}$$

with $\Gamma(\cdot)$ being the Gamma function. When $\alpha\to 1$, the problem (1.2) reduces to the classical parabolic SPDE. The deterministic version of (1.2) occurs in a wide range of applications. For example, it is used to describe diffusion in media with fractal geometry [23], relaxation phenomena in complex viscoelastic materials [9], a non-Markovian diffusion process with a memory [22] and so on. In addition, the numerical study of such a deterministic problem can be found in a lot of literature, e.g., [13, 16, 18–20, 32].

With $D_t^{1-\alpha}$ replaced by the left-sided Caputo fractional derivative $\partial_t^{1-\alpha}$:

$$\partial_t^{1-\alpha}u(t)=\begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\frac{\partial}{\partial s}u(s)\,\mathrm{d}s, & \alpha\in(0,1), \\ \frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\frac{\partial}{\partial s}u(s)\,\mathrm{d}s, & \alpha\in(1,2), \end{cases} \tag{1.4}$$

the problem (1.2) becomes another class of important fractional SPDEs:

$$\begin{aligned} &\mathrm{d}u(t)+A\partial_t^{1-\alpha}u(t)\,\mathrm{d}t=\mathrm{d}W_Q^H(t) \quad \text{for } t\in(0,\ T] \\ &\quad \text{with } u(0,\cdot)=u_0(\cdot) \text{ and } 0<\alpha<1. \end{aligned} \tag{1.5}$$

For (1.5) with $\alpha\in(1,2)$ and $H=1/2$, strong and weak approximation errors of finite element discretizations have been rigorously analyzed in [14] and [15], respectively. Later on, for a space-time white noise, sharper strong convergence rates of the time discretization and of the full-discrete finite element method are shown in [11] and [10], respectively. However, we are not aware of any rigorous numerical analysis of (1.1) with fBm. This article aims to fill the gap and to investigate the regularity properties and strong approximations of SPVE (1.1).

In [7], the temporal regularity of stochastic convolutions arising in the mild solution of (1.1) perturbed by a white noise was studied. A general version of (1.1) was considered in [33] and the existence and regularity results were established based on the explicit formula for the scalar resolvent function and the properties of Mittage-Leffler's function. Sperlich [25] derived the optimal conditions for the existence of a unique mild solution of