Comparison Principles for Some Fully Nonlinear Sub-Elliptic Equations on the Heisenberg Group

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Abstract. In this paper, we prove a form of the strong comparison principle for a class of fully nonlinear subelliptic operators of the form $\nabla_{H,s}^2 \psi + L(\cdot, \psi, \nabla_H \psi)$ on the Heisenberg group, which include the CR invariant operators.

Key Words: Comparison principle, subellipticity, CR invariance, Heisenberg group, propagation of touching points.

AMS Subject Classifications: 35J60, 35J70, 35B51, 35B65, 35D40, 53C21, 58J70

1 Introduction

In this paper, we establish a form of the comparison principle for a class of subelliptic equations on the Heisenberg group.

Let Ω be an open connected subset of \mathbb{R}^n ($n \ge 1$), the *n*-dimensional Euclidean space. Assume that $u, v \in C^2(\Omega)$ satisfy

$$u \ge v \quad \text{in} \quad \Omega. \tag{1.1}$$

The standard form of the strong comparison principle for nonlinear second order elliptic operators $F(x, u, \nabla u, \nabla^2 u)$ is the following. Here F(x, s, p, M) is of class C^1 , $x \in \Omega$, $s \in \mathbb{R}^1$, $p \in \mathbb{R}^n$, $M \in S^{n \times n}$, the set of all $n \times n$ real symmetric matrices, and is elliptic, i.e.,

$$\frac{\partial F}{\partial M_{ij}} > 0.$$

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Strong comparison principle. Let $u, v \in C^2(\Omega)$ satisfy (1.1) and

$$F(x,u,\nabla u,\nabla^2 u) \leq F(x,v,\nabla v,\nabla^2 v)$$
 in Ω

Then we have

either
$$u > v$$
 in Ω , or $u \equiv v$ in Ω .

In [9] and [10], L. Caffarelli, L. Nirenberg and the first named author obtained some forms of the comparison principle for singular solutions of the nonlinear elliptic operators of the form $F(x,u,\nabla u,\nabla^2 u)$. Recently, the strong comparison principle and the Hopf Lemma was established for viscosity solutions to the equations of the form $F(x,u,\nabla u,\nabla^2 u) = 1$ when one of the competitors is $C^{1,1}$ by Y. Y. Li, L. Nguyen and B. Wang in [33].

In recent years, comparison principles for degenerate elliptic equations have been widely studied; see [1-7, 14-37] and the references therein. One type of those equations, which appeared in [9-12,18,19,29,30,34,35], involve a symmetric matrix function

$$G[u] := \nabla^2 u + L(x, u, \nabla u), \qquad (1.2)$$

where $L \in C^{0,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, is in $S^{n \times n}$.

One such matrix operator is the conformal Hessian matrix operator (see e.g., [28, 38] and the references therein), i.e.,

$$H[u] = \nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^2 I_n,$$

where I_n denotes the $n \times n$ identity matrix, and, for $p, q \in \mathbb{R}^n$, $p \otimes q$ denotes the $n \times n$ matrix with entries $(p \otimes q)_{ij} = p_i q_j$ for $i, j = 1, \dots, n$.

Let *U* be an open subset of $S^{n \times n}$ satisfying

$$0 \in \partial U$$
, $U + \mathcal{P} \subset U$, $O^t U O \subset U$, $\forall O \in O(n)$ $t U \subset U$, $\forall t > 0$,

where \mathcal{P} denotes the set of all $n \times n$ non-negative real matrices and O(n) denotes the set of all $n \times n$ real orthogonal matrices.

In [34], it was shown that, under the assumption

$$\operatorname{diag}\{1,0,\cdots,0\} \in \partial U,\tag{1.3}$$

the strong comparison principle and Hopf Lemma fail for a class of equations of the form

$$G[u] \in \partial U.$$

Conversely, if (1.3) does not hold, then the strong comparison principle and Hopf Lemma holds.

Although the strong comparison principle fails under assumption (1.3), the first named author proved that a weak form of strong comparison principle holds for the