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Two Kinds of New Energy-Preserving Schemes for the Coupled Nonlinear Schrödinger Equations

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Abstract. In this paper, we mainly propose two kinds of high-accuracy schemes for the coupled nonlinear Schrödinger (CNLS) equations, based on the Fourier pseudospectral method (FPM), the high-order compact method (HOCM) and the Hamiltonian boundary value methods (HBVMs). With periodic boundary conditions, the proposed schemes admit the global energy conservation law and converge with even-order accuracy in time. Numerical results are presented to demonstrate the accuracy, energy-preserving and long-time numerical behaviors. Compared with symplectic Runge-Kutta methods (SRKMs), the proposed schemes are assuredly more effective to preserve energy, which is consistent with our theoretical analysis.

AMS subject classifications: 37M05, 65M06

Key words: Hamiltonian boundary value methods, Fourier pseudospectral method, high-order compact method, coupled nonlinear Schrödinger equations.

1 Introduction

Since the explosion of interest in nonlinear science, the nonlinear Schrödinger equation plays a central role in a wide range of physical phenomena, including nonlinear optics [19], plasma physics [23], atomic Bose-Einstein condensates [2], etc. In order to describe two interacting nonlinear packets in dispersive or conservative systems, in 1967, the coupled nonlinear Schrödinger (CNLS) equations were firstly derived by Benney and Newell [1].

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In this paper, we are mainly concerned with the CNLS equations

$$\begin{cases} \mathrm{i}\phi_t+\phi_{xx}+(|\phi|^2+\beta|\psi|^2)\phi=0, \\ \mathrm{i}\psi_t+\psi_{xx}+(|\psi|^2+\beta|\phi|^2)\psi=0, \end{cases} \tag{1.1}$$

with suitable initial data and periodic boundary conditions. Here, $\phi(x,t)$ and $\psi(x,t)$, $(x,t)\in[x_L,x_R]\times[0,T]$, are complex envelopes of two wave packets, β is the coupling constant and i is the imaginary unit. Due to the intrinsic stability of equations, solitons could be formed when the nonlinear term exactly balances the wave packet dispersion and their dynamics have appeared in many important applications [27, 29]. Therefore, the signification about solitons of the CNLS equations (1.1) has been widely acknowledged.

So far, numerous researches have been conducted to solve the CNLS equations. Based on the finite difference method, Ismail and Alamri have achieved linearly implicit conservative scheme [20] and fourth-order explicit scheme [21]. Both of them could preserve discrete energy exactly. Kurtinaitis and Ivanauska [24] employed explicit, implicit, Crank-Nicolson type and Hopsotch type finite difference scheme, respectively, to simulate the dynamics of the 3-CNLS equations. In [22], the Galerkin method was utilized for the CNLS equations. In [4], the authors discussed how to apply the differential transformation method to solve the CNLS equations. Furthermore, Bao et al. [3] simulated the bright and dark soliton solutions of the CNLS system successfully by using the time-splitting pseudospectral method.

In recent decades, it has been widely convinced that structure-preserving methods, which are able to preserve the intrinsic properties of the original system, could achieve long-time high precision simulation in most cases. Therefore, many important results were subsequently reported, such as symplectic Runge-Kutta methods (SRKMs) [18, 31], multi-symplectic methods [15, 28] and semi-explicit and explicit multi-symplectic methods [30,32]. In addition, energy-preserving methods, including discrete gradient method [26], local energy-preserving method [17], average vector field method [16] and Hamiltonian boundary value methods (HBVMs), were widely applied for numerical simulation as well. Due to its remarkable energy-preserving property, the HBVMs has attracted much attention since it was first proposed for ODEs by Brugnano et al. [5, 6] in 2010. In [7], the efficient implementation of the HBVMs was fully discussed. Then Brugnano and Sun [8] further proposed a multiple invariants conserving method for Hamiltonian ODEs. Recently, this method was generalised to solve semilinear wave equations [9], nonlinear Schrödinger equation [12, 13] and other Hamiltonian PDEs [10, 11]. To the best of our knowledge, there exists no report about the application of the HBVMs for the CNLS equations at the moment, so we combine this energy-preserving method with the Fourier pseudospectral method (FPM) [14] and the high-order compact method (HOCM) [25] to construct two kinds of numerical schemes.

As is well known, we can rewrite the CNLS equations (1.1) as the infinite-dimensional Hamiltonian system

$$\frac{\partial z}{\partial t}=J\frac{\delta\mathcal{H}}{\delta z}, \tag{1.2}$$