Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2017-0204

Vol. **25**, No. 2, pp. 532-563
February 2019

Conservative and Dissipative Local Discontinuous Galerkin Methods for Korteweg-de Vries Type Equations

Qian Zhang[1] and Yinhua Xia[1,*]

[1] *School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China.*

Received 24 September 2017; Accepted (in revised version) 12 February 2018

Abstract. In this paper, we develop the Hamiltonian conservative and L^2 conservative local discontinuous Galerkin (LDG) schemes for the Korteweg-de Vries (KdV) type equations with the minimal stencil. For the time discretization, we adopt the semi-implicit spectral deferred correction (SDC) method to achieve the high order accuracy and efficiency. Also we compare the schemes with the dissipative LDG scheme. Stability of the fully discrete schemes is provided by Fourier analysis for the linearized KdV equation. Numerical examples are shown to illustrate the capability of these schemes. Compared with the dissipative LDG scheme, the numerical simulations also indicate that the conservative LDG scheme with high order time discretization can reduce the long time phase error validly.

AMS subject classifications: 65M60, 65M12, 35Q53

Key words: Local discontinuous Galerkin method, conservative and dissipative schemes, Korteweg-de Vries type equations, semi-implicit spectral deferred correction method.

---

1 Introduction

In this paper, we consider the initial value problem of Korteweg-de Vries (KdV) equation

$$\begin{cases} u_t+f(u)_x+\varepsilon u_{xxx}=0, & x\in I=[a,b], \quad t>0, \\ u(x,0)=u_0(x), \end{cases} \tag{1.1}$$

where t is time, x is the space coordinate in the direction of propagation, $a,b,\varepsilon\in\mathbb{R},\varepsilon>0$. With smooth enough initial condition $u_0(x)$, we can obtain the existence and uniqueness of solution [8]. The KdV equation is first introduced by Boussinesq (1877) and rediscovered by Diederik Korteweg and Gustav de Vries in 1895 [14], in which studies the

*Corresponding author. *Email addresses:* `gelee@mail.ustc.edu.cn` (Q. Zhang), `yhxia@ustc.edu.cn` (Y. Xia)

small-amplitude long waves in shallow water. In the study of water wave, it has two well-known solutions, cnoidal wave and solitary wave solutions. In the last decades, since the soliton solution proposed by Zabusky and Kruskal [33], the KdV equation has risen a considerable interest by physicists and mathematicians. Actually, the KdV equation is a mathematical model for the propagation of nonlinear dispersive long waves in many branches of physics and engineering including aerology, oceanography, plasma physic, geology, among many others.

Various numerical methods of solving this equation have been proposed, like finite-difference schemes [12, 20], pseudospectral methods [10], heat balance integral method [15] and finite element method, especially discontinuous Galerkin method. The discontinuous Galerkin method (DG method) is a class of finite element methods using completely discontinuous piecewise polynomial functions as numerical approximation and test functions. The DG method was first introduced in 1973 by Reed and Hill in [19] for solving steady state linear hyperbolic equations. The important ingredient of this method is the design of suitable inter-element boundary treatments (so called numerical fluxes) to obtain highly accurate and stable schemes in many situations.

Within the DG framework, the method was extended to deal with derivatives of order higher than one, i.e. local discontinuous Galerkin (LDG) method. The first LDG method was introduced by Cockburn and Shu in [5] for solving convection-diffusion equation. Their work was motivated by the successful numerical experiments of Bassi and Rebay [3] for compressible Navier-Stokes equations. Later, Yan and Shu developed a LDG numerical method for a general KdV type equation containing third order derivatives in [31], and they generalized the LDG method to PDEs with fourth and fifth spatial derivatives in [32]. Levy, Shu and Yan [17] developed LDG methods for nonlinear dispersive equations that have compactly supported traveling wave solutions, the so-called compactons. More recently, Xu and Shu further generalized the LDG method to solve a series of nonlinear wave equations [24–27]. We refer to the review paper [29] of LDG methods for high-order time-dependent partial differential equations.

According to the selection of numerical flux function for the nonlinear term $f(u)$ and the dispersive term εu_{xxx} in KdV equations, the DG method can be divided into dissipative and conservative schemes. Conservative discretization scheme means that this scheme can preserve certain conserved quantities discretely. In the numerical experiments of [4], the higher accuracy and better stability of the conservative scheme over long temporal intervals can be seen. Usually, the conservation of L^2 energy

$$H_1=\int \frac{1}{2}u^2dx, \tag{1.2}$$

and the conservation of the Hamiltonian

$$H_2=\int \frac{\varepsilon}{2}u_x^2-V(u)dx, \quad V(u)=\int^u f(\zeta)\, d\zeta, \tag{1.3}$$

are considered, since the KdV equation is a Hamiltonian system [11].