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Exact Boundary Controllability on a Planar Tree-Like Network of Vibrating Strings with Dynamical Boundary Conditions

Tatsien Li[1,*] and Yue Wang[2]

[1] *School of Mathematical Sciences, Fudan University & Shanghai Key Laboratory for Contemporary Applied Mathematic & Nonlinear Mathematical Modeling and Methods Laboratory, Shanghai 200433, P. R. China;*
[2] *School of Mathematical Sciences, Fudan University, Shanghai 200433, P. R. China.*

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Abstract. This paper concerns a planar tree-like network of vibrating strings with point masses at the nodes. We use a constructive method with modular structure to get the 'one-sided' exact boundary controllability for this system with dynamical boundary conditions. Moreover, by constructing the 'longest' chain-like subnetwork and its 'midpoint', we divide the whole tree-like network into two tree-like sub-networks, and prove the 'two-sided' exact boundary controllability.

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Key words: A planar tree-like network of vibrating strings, dynamical boundary condition, exact boundary controllability.

1 Introduction

On a planar tree-like network of vibrating strings, the exact boundary controllability for linear wave equations with Dirichlet boundary conditions was studied in [1]- [2] and [8]. In the quasilinear case, by establishing the semi-global piecewise C^2 solution for quasilinear wave equations with Dirichlet, Neumann, Robin and dissipative boundary conditions on a planar tree-like network of strings, Gu & Li ([3]) got the 'one-sided' exact boundary controllability, respectively. Thus, if the network has k simple nodes, then the 'one-sided' exact boundary controllability can be obtained by means of boundary controls acting on $(k-1)$ simple nodes.

For 1-D elastic strings with tip-masses at the ends, based on the theory of semi-global C^2 solution to the corresponding mixed problem, we have established the local exact

*Corresponding author. *Email addresses:* dqli@fudan.edu.cn (T. Li), yue_wang@fudan.edu.cn (Y. Wang)

boundary controllability for (a coupled system of) quasilinear wave equations with dynamical boundary conditions in [9, 10] by means of a constructive method with modular schedule (see, e.g., [5, 6]).

In this paper, we consider a planar tree-like network of vibrating strings with point masses at the nodes. Dynamical boundary conditions are encountered in analyzing mechanical behavior of point masses at the simple nodes. While, on the multiple nodes, through the continuity of displacements and the corresponding total stress boundary condition, the interface conditions are obtained. For a tree-like network with k simple nodes, we get its 'one-sided' exact boundary controllability by only $(k-1)$ controls given on simple nodes, which is consistent with the result without point masses in [3]. Moreover, this paper examines the 'two-sided' exact boundary controllability for the first time, that is, the controls are given on all the simple nodes. This result not only leads to a significant reduction of controllability time, but also applies to the case without point masses at nodes, which can be regarded as an important complement to the results in [3].

This paper is organized as follows. The mathematical model and main results of 'one-sided' and 'two-sided' exact boundary controllability on a planar tree-like network are presented in Section 2. To prove these results, in Section 3, we first establish the existence and uniqueness of semi-global piecewise C^2 solution to the corresponding mixed problem on a planar tree-like network. Then based on this, we prove the 'one-sided' exact boundary controllability for this system in Section 4 by constructing the semi-global piecewise C^2 solution, which satisfies simultaneously the initial condition, the final condition, all the given boundary conditions and all the interface conditions. Then, in Section 5, we construct the 'longest' chain-like subnetwork and its 'midpoint', and divide the whole tree-like network into two tree-like sub-networks artificially, then finish the proof of the 'two-sided' boundary controllability.

2 Modeling and main results

In this section, we consider a planar tree-like network which is composed of $N(N>1)$ vibrating strings: $S_1,...,S_N$. Without loss of generality, we suppose that one end of string S_1 is a simple node. We take this simple node as the starting node E of the network. For string S_i, let node d_{i0}(node d_{10} is just E) and node d_{i1} be its two ends, the $x-$ coordinates of which are $x=d_{i0}$ and $x=d_{i1}$, respectively. And $L_i=d_{i1}-d_{i0}$ is the length of string S_i. We always suppose that node d_{i0} is closer to E than node d_{i1} in the network, and all nodes are with unit mass (See Figure 1). The case without masses at nodes can be treated in a similar and simpler way.

Let $\mathcal{M}$ and $\mathcal{S}$ be two subsets of $\{1,...N\}$, $\mathcal{M}\cup\mathcal{S}=\{1,...N\}$. $i\in\mathcal{M}$ if and only if d_{i1} is a multiple node, while, $i\in\mathcal{S}$ if and only if d_{i1} is a simple node. Thus, the collection of simple nodes on the network consists of E and $\{d_{i1},i\in\mathcal{S}\}$, while, the set of multiple nodes is $\{d_{i1},i\in\mathcal{M}\}$.

For $i=1,...,N$, we consider the following quasilinear wave equations on the string S_i