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Hamiltonian Boundary Value Method for the Nonlinear Schrödinger Equation and the Korteweg-de Vries Equation

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Abstract. In this paper, we introduce the Hamiltonian boundary value method (HBVM) to solve nonlinear Hamiltonian PDEs. We use the idea of Fourier pseudospectral method in spatial direction, which leads to the finite-dimensional Hamiltonian system. The HBVM, which can preserve the Hamiltonian effectively, is applied in time direction. Then the nonlinear Schrödinger (NLS) equation and the Korteweg-de Vries (KdV) equation are taken as examples to show the validity of the proposed method. Numerical results confirm that the proposed method can simulate the propagation and collision of different solitons well. Meanwhile the corresponding errors in Hamiltonian and other intrinsic invariants are presented to show the good preservation property of the proposed method during long-time numerical calculation.

AMS subject classifications: 65M10, 78A48

Key words: Hamiltonian boundary value method, Hamiltonian-preserving, nonlinear Schrödinger equation, Korteweg-de Vries equation.

1 Introduction

In recent years, there exists an increasing emphasis on constructing numerical methods that can preserve structural properties of the continuous Hamiltonian systems. These methods are called geometric numerical integrators or structure-preserving numerical methods in general. Structure-preserving numerical methods originate from the numerical methods for ODEs and have a huge growth in the last decades, such as, symplectic method [1, 2], discrete gradient method [3], average vector field (AVF) [4], the HBVM [5, 6] and so on. Then people naturally pay more attention to generalising these

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methods to PDEs, meanwhile, keeping geometric properties of original equations. So far, there are two main branches to preserve structures of Hamiltonian PDEs. The first is the multi-symplectic method (MSM) [7–11], which can preserve the multi-symplectic structure of the Hamiltonian system exactly. The second is Hamiltonian-preserving methods or energy-preserving methods including discrete variational derivative method [12, 13], continuous stage Runge-Kutta method [14, 15], the AVF [16–18], etc.. At present, it is widely acknowledged that Hamiltonian-preserving, as intrinsic geometric property, is of much significance during numerical simulations.

In this paper, we consider Hamiltonian system [16]

$$\frac{\partial u}{\partial t}=J\frac{\delta\mathcal{H}}{\delta u} \tag{1.1}$$

in the domain $\Omega=(x,t)\in\mathbf{R}\times\mathbf{R}$, where x and t denote space and time variables, respectively. Here, J is a constant linear skew-symmetric operator and the Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{H}[u]=\int_\Omega H(x;u,u_x,u_{xx},\cdots)dx$$

and the variational derivative [16] is given by

$$\frac{\delta\mathcal{H}}{\delta u}=\frac{\partial H}{\partial u}-\partial_x\left(\frac{\partial H}{\partial u_x}\right)+\partial_x^2\left(\frac{\partial H}{\partial u_{xx}}\right)-\cdots.$$

According to the form of (1.1), we have

$$\frac{\partial\mathcal{H}}{\partial t}=\frac{\delta\mathcal{H}}{\delta u}\frac{\partial u}{\partial t}=\frac{\delta\mathcal{H}}{\delta u}J\frac{\delta\mathcal{H}}{\delta u}=0. \tag{1.2}$$

Obviously, the Hamiltonian is an invariant. Sometimes, Hamiltonian system can be written as

$$\frac{\partial u}{\partial t}=\mathcal{J}(u)\frac{\delta\mathcal{H}}{\delta u}, \tag{1.3}$$

where $\mathcal{J}(u)$ is a skew-symmetric operator which depends on the solution $u(x,t)$. It's easy to know, similar result can be obtain for the form of (1.3).

The HBVM was first derived for ODEs by Brugnano et al. [5] in 2009. This method has attracted much attention in recent years because of its remarkable Hamiltonian-preserving property under appropriate discretization for ODEs. In 2014, Brugnano and Sun [21] proposed a multiple invariants conserving method for Hamiltonian ODEs. Then Brugnano [20] generalised the HBVM to solve semilinear wave equation. To the best of our knowledge, this method has few applications in general nonlinear Hamiltonian PDEs and comparisons with other Hamiltonian-preserving methods. So we apply this method for the NLS and KdV equation [24–27] to show its numerical characters.

With this premise, the rest of the paper is arranged as follows. In Section 2, we review the Fourier pseudospectral method [22–24] for space-discretization and transform