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Mathematical Models for the Propagation of Stress Waves in Elastic Rods: Exact Solutions and Numerical Simulation

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Abstract. In this work, the Bishop and Love models for longitudinal vibrations are adopted to study the dynamics of isotropic rods with conical and exponential cross-sections. Exact solutions of both models are derived, using appropriate transformations. The analytical solutions of these two models are obtained in terms of generalised hypergeometric functions and Legendre spherical functions respectively. The exact solution of Love model for a rod with exponential cross-section is expressed as a sum of Gauss hypergeometric functions. The models are solved numerically by using the method of lines to reduce the original PDE to a system of ODEs. The accuracy of the numerical approximations is studied in the case of special solutions.

AMS subject classifications: 33C90, 35Q74, 35L35

Key words: Longitudinal vibration of rods, variable cross-section, exact solution, method of lines, hypergeometric functions.

1 Introduction

Mathematical modelling of wave propagation in elastic rods has attracted considerable interest from both scientific and industrial communities due to the increasingly complexity of the geometry of the new waveguides and the physical phenomenon itself. Because of this fact many models and numerical methods, designed to ease, the analysis of the rod behaviour while experiencing wave propagation, have been reported in literature. Green [1] and Graff [2] have reviewed most of the developments in the theoretical study

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of the dynamics of acoustic waveguides and the vibrating rod with constant geometry before 1970. The need for useful analytical results for longitudinal vibrations in rods with more complex geometries has motivated the work by Eisenberger on the vibration of tapered structure in which, he showed that the natural frequencies were only modified slightly by the gradual diminution of the thickness [3]. Following on the results of Abrate [4] and Bapat [5] on the derivation of the exact solutions for longitudinal vibration of conical and catenoidal rods, Kumar and Sujith have presented their approach for obtaining the analytical solution for axial vibration of sinusoidal and polynomial cross-section rods [6]. It is important to emphasise that the results obtained by these authors are based on the classical wave equation with variable coefficients.

However many mathematical models of higher order, describing longitudinal wave propagation in elastic solids have been derived in order to analyse the effect of different materials and geometries on the vibration characteristics, thus avoiding costly experimental studies [7–9]. These models are too complicated to solve analytically due to the complexity of the boundary conditions, of the geometries and the heterogeneity of the media. To overcome those difficulties, numerical methods are applied. However, while numerical methods provide approximations to exact solutions, they seldom provide adequate insight into their properties. Hence the motivation for deriving exact solutions in explicit form is twofold: firstly providing insight into the physics of the problem; and secondly establishing and benchmarking the accuracy of the numerical solutions. Efforts to derive analytical solutions for intricate models are often limited by the lack of appropriate mathematical tools.

In this paper, the Love and Bishop models for longitudinal vibrations are applied to study the dynamics of isotropic rods with conical and exponential cross-sections. The choice of the conical or exponential shape is justified by the fact that the methods for modelling waveguides with complex geometry consist of dividing them into cylindrical and conical/exponential elements which make the entire waveguide. As a result, the cross-section radius is position-dependent and causes the geometric parameters of the rod to be variable [10]. A new approach for the derivation of the exact solutions of the obtained models is presented in this paper. The analytical solutions of these two models for a conical rod are obtained in terms of generalized hypergeometric and Legendre spherical functions. The exact solution of the Love model for a rod with exponential cross-section is expressed in terms of the Gauss hypergeometric functions. Recall that the Gauss hypergeometric functions are special functions containing as special cases, the majority of the commonly used functions in Analysis [11–13].

Numerical solutions are obtained by the method of lines which reduces the partial differential equations describing the dynamics of the Love rod to a system of ordinary differential equations [14, 15]. The reliability of numerical approximation is tested using the exact solutions, obtained for some special cases. More precisely for testing of accuracy of the numerical solution we chose special initial conditions, namely those obtained for initial longitudinal displacements of the rod that are proportional to one of the eigenfunctions of the system and zero initial velocities. In this case the vibrations of every