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A Second-Order Finite Difference Method for Two-Dimensional Fractional Percolation Equations

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Abstract. A finite difference method which is second-order accurate in time and in space is proposed for two-dimensional fractional percolation equations. Using the Fourier transform, a general approximation for the mixed fractional derivatives is analyzed. An approach based on the classical Crank-Nicolson scheme combined with the Richardson extrapolation is used to obtain temporally and spatially second-order accurate numerical estimates. Consistency, stability and convergence of the method are established. Numerical experiments illustrating the effectiveness of the theoretical analysis are provided.

AMS subject classifications: 65M06, 26A33, 35R11

Key words: Fractional percolation equation, second-order finite difference method, Crank-Nicolson scheme, Richardson extrapolation, stability analysis.

1 Introduction

In this paper, we are concerned with the development of finite difference methods for the two-dimensional fractional percolation problem which seeks unknown pressure function $p(x,y,t)$ satisfying

$$\frac{\partial p}{\partial t}=\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\left(k_x(x,y)\frac{\partial^{\alpha_1}p}{\partial x^{\alpha_1}}\right)+\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}\left(k_y(x,y)\frac{\partial^{\alpha_2}p}{\partial y^{\alpha_2}}\right)+f(x,y,t),\quad (x,y)\in\Omega,\quad 0<t\le T, \tag{1.1}$$

$$p(x,y,0)=p_0(x,y),\qquad (x,y)\in\Omega, \tag{1.2}$$

$$p(x_L,y,t)=p(x,y_L,t)=0, \tag{1.3}$$

$$p(x_R,y,t)=v_x(y,t),\quad p(x,y_R,t)=v_y(x,t),\qquad (x,y)\in\Omega,\quad 0\le t\le T,$$

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where $\Omega=\{(x,y)|x_L\le x\le x_R,y_L\le y\le y_R\}$,$0<\alpha_1,\alpha_2<1$,$0<\beta_1,\beta_2\le 1$, $f(x,y,t)$ is the source term, positive $k_x(x,y)$ and $k_y(x,y)$ are percolation coefficients along the x and y direction, respectively.

The fractional partial derivatives in (1.1) are defined in the Riemann-Liouville form. Generally, for any $\gamma>0$, the Riemann-Liouville fractional partial derivatives $\frac{\partial^\gamma w(x,y)}{\partial x^\gamma}$ and $\frac{\partial^\gamma w(x,y)}{\partial y^\gamma}$ of order γ are defined by [28,32,35]

$$\frac{\partial^\gamma w(x,y)}{\partial x^\gamma}=\frac{1}{\Gamma(n-\gamma)}\frac{\partial^n}{\partial x^n}\int_{x_L}^{x}\frac{w(\xi,y)}{(x-\xi)^{\gamma+1-n}}d\xi \tag{1.4}$$

and

$$\frac{\partial^\gamma w(x,y)}{\partial y^\gamma}=\frac{1}{\Gamma(n-\gamma)}\frac{\partial^n}{\partial y^n}\int_{y_L}^{y}\frac{w(x,\eta)}{(y-\eta)^{\gamma+1-n}}d\eta, \tag{1.5}$$

where n is an integer such that $n-1<\gamma\le n$. If γ is an integer, then the above definitions give the standard integer partial derivatives.

The percolation equations have been applied successfully in groundwater hydraulics, groundwater dynamics and fluid dynamics in porous media [7, 31, 40]. Under the assumptions of seepage flow continuity and the traditional Darcy's law, the traditional percolation equation [24, 34] for two-dimensional seepage flow in porous media is just the special case of equation (1.1) that $\alpha_i=\beta_i=1$ for $i=1,2$. In view of the limitations of these two assumptions above, He [12] proposed the modified Darcy's law with Riemann-Liouville fractional derivatives

$$q_x=k_x(x,y)\frac{\partial^{\alpha_1}p}{\partial x^{\alpha_1}},\qquad q_y=k_y(x,y)\frac{\partial^{\alpha_2}p}{\partial y^{\alpha_2}},\qquad 0<\alpha_1,\alpha_2<1, \tag{1.6}$$

as a generalization of Darcy's law for realistically describing the movement of solute in a non-homogeneous porous medium. Furthermore, considering the fact that the seepage flow is neither continued nor rigid body motion, He employed the fractional differential operators $\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}$ and $\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}$ where $0<\beta_1,\beta_2\le 1$ in the percolation equation, and then the fractional percolation model (1.1) was obtained, which is the focus of this paper.

As analytic solutions of most fractional differential equations cannot be obtained explicitly, numerical methods become major ways and a wide variety of techniques have been developed, including finite difference methods [25–27, 29, 37–39, 49–51], finite element methods [2,8–10,16], finite volume methods [42,44,45,48], spectral methods [15,17, 47], and mesh-free methods [11,20]. Recently, Liu et al. [21] proposed a first-order alternating direction implicit scheme for the three-dimensional non-continued seepage flow in uniform media and a second-order method which combined modified Douglas scheme with Richardson extrapolation for the three-dimensional continued seepage flow in non-uniform media. Chen et al. [5] developed an implicit finite difference method for the initial-boundary value problem of one-dimensional fractional percolation equation with left-sided mixed Riemann-Liouville fractional derivative, and they [6] considered an alternating direction implicit difference method for the two-dimensional case by a similar