Monotone Finite Difference Schemes for Anisotropic Diffusion Problems via Nonnegative Directional Splittings

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Abstract. Nonnegative directional splittings of anisotropic diffusion operators in the divergence form are investigated. Conditions are established for nonnegative directional splittings to hold in a neighborhood of an arbitrary interior point. The result is used to construct monotone finite difference schemes for the boundary value problem of anisotropic diffusion operators. It is shown that such a monotone scheme can be constructed if the underlying diffusion matrix is continuous on the closure of the physical domain and symmetric and uniformly positive definite on the domain, the mesh spacing is sufficiently small, and the size of finite difference stencil is sufficiently large. An upper bound for the stencil size is obtained, which is determined completely by the diffusion matrix. Loosely speaking, the more anisotropic the diffusion matrix is, the larger stencil is required. An exception is the situation with a strictly diagonally dominant diffusion matrix where a three-by-three stencil is sufficient for the construction of a monotone finite difference scheme. Numerical examples are presented to illustrate the theoretical findings.

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1 Introduction

We consider the finite difference solution of the boundary value problem

$$\begin{cases} -\nabla \cdot (\mathbb{D}\nabla u) = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = (0,1) \times (0,1)$, f and g are given functions, and the diffusion matrix,

$$\mathbb{D} = \begin{bmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{bmatrix}, \tag{1.2}$$

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is assumed to be continuous on $\overline{\Omega}\equiv\Omega\cup\partial\Omega$ and symmetric and uniformly positive definite on Ω. We are interested in the case where $\mathbb{D}$ depends on spatial location (heterogeneous diffusion) and has unequal eigenvalues at least on some portion of Ω (anisotropic diffusion). For this case, (1.1) is often called a heterogeneous anisotropic diffusion problem. Anisotropic diffusion arises from various areas of science and engineering, including plasma physics [9], petroleum reservoir simulation [7,20], and image processing [12,31].

When a conventional method such as a finite element, a finite difference, or a finite volume method, is applied to this problem, spurious oscillations can occur in the numerical solution. This difficulty can be overcome by numerical schemes satisfying a discrete maximum principle (DMP). The development and studies of DMP-preserving schemes have received considerable attention in the past particularly in the context of finite element and finite volume methods, e.g., see [3–6, 9–11, 13–19, 24–30, 32, 33]. On the other hand, relatively less work has been done in the context of finite difference (FD) methods. The major effort has been made to study monotone (or called positive-type or nonnegative-type) schemes, which belong to a special class of DMP-preserving schemes with the coefficient matrix (or the Jacobian matrix in the nonlinear case) of the corresponding algebraic system being an *M*-matrix (e.g., see [30] for the definition of *M*-matrices). For example, Motzkin and Wasow [21] prove that a monotone FD scheme exists for any linear second-order elliptic problem when the mesh is sufficiently fine. Greenspan and Jain [8], using nonnegative directional splittings (see the definition below), propose a way to construct such schemes for elliptic operators in the nondivergence form

$$L[u]\equiv a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy} \quad \text{with} \quad b(x,y)^2<a(x,y)c(x,y). \qquad (1.3)$$

The results are extended to elliptic problems in the divergence form (1.1) by Weickert [31]. Consistent and stable monotone FD schemes are shown to be convergent (to the solution) for linear second-order elliptic problems by Bramble et al. [2] and (to the viscosity solution) for nonlinear second-order degenerate elliptic or parabolic partial differential equations by Barles and Souganidis [1]. Oberman [23] studies degenerate elliptic schemes (a special type of monotone scheme) for a general class of nonlinear degenerate elliptic problems.

The objective of this work is to study the construction of monotone FD schemes for problems in the form of (1.1) using nonnegative directional splittings. Weickert's results are improved in two aspects. We first present a condition under which nonnegative directional splittings hold for a neighborhood of an arbitrary point. As we will see below, it is necessary to consider nonnegative directional splittings in a neighborhood for the construction of monotone FD schemes. We then extend the result to the situation where the coefficient $b(x,y)$ can change sign over the domain. To be more specific, we recall that Weickert [31] considers the directional splitting

$$\nabla\cdot(\mathbb{D}\nabla u)(x_0,y_0)=\partial_x(\gamma_0\partial_x u)(x_0,y_0)+\partial_\beta(\gamma_1\partial_\beta u)(x_0,y_0)+\partial_y(\gamma_2\partial_y u)(x_0,y_0), \qquad (1.4)$$