Commun. Comput. Phys.
doi: 10.4208/cicp.281114.290615a

Vol. **18**, No. 5, pp. 1336-1351
November 2015

Laplacian Preconditioning for the Inverse Arnoldi Method

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Received 28 November 2014; Accepted (in revised version) 29 June 2015

Abstract. Many physical processes are described by elliptic or parabolic partial differential equations. For linear stability problems associated with such equations, the inverse Laplacian provides a very effective preconditioner. In addition, it is also readily available in most scientific calculations in the form of a Poisson solver or an implicit diffusive timestep. We incorporate Laplacian preconditioning into the inverse Arnoldi method, using BiCGSTAB to solve the large linear systems. Two successful implementations are described: spherical Couette flow described by the Navier-Stokes equations and Bose-Einstein condensation described by the nonlinear Schrödinger equation.

AMS subject classifications: 37M20, 65F08, 65F18, 65P10, 65P30, 76E07
Key words: To be provided by author

1 Introduction

Many physical systems are governed by parabolic evolution equations of the general form

$$\partial_t U = LU + N(U), \tag{1.1}$$

where L is the Laplacian operator and N represents some combination of nonlinear terms or a multiplicative potential. Two examples which we will consider are the Navier-Stokes equations

$$\partial_t U = -(U\cdot\nabla)U - \nabla P + \nu\nabla^2 U, \tag{1.2a}$$
$$\nabla\cdot U = 0, \tag{1.2b}$$

and the nonlinear Schrödinger equation

$$-i\partial_t \Psi = \left[\frac{1}{2}\nabla^2 + \mu - V(\mathbf{x}) - a|\Psi|^2\right]\Psi. \tag{1.3}$$

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Steady solutions of (1.1) satisfy

$$0 = LU + N(U) \tag{1.4}$$

and the Jacobian operator evaluated at U is defined by

$$A \equiv L + N_U, \tag{1.5}$$

where N_U is the linearization of N at U. Steady bifurcations from U occur when an eigenvalue of A crosses zero. For this reason, we are interested in the eigenvalues of (1.5) which are closest to zero. These eigenvalues can be calculated by the classic inverse power method, generalized to the inverse Arnoldi method [1]. The sequence $\{u_k \equiv A^{-(k-1)} u_1;\, k = 1, \cdots K\}$ is generated by solving

$$A u_{k+1} = u_k. \tag{1.6}$$

This sequence is orthonormalized by the usual Arnoldi process to yield the basis $\{v_k\}$ for the Krylov space and the upper Hessenberg matrix

$$H_{jk} \equiv \langle v_j, A^{-1} v_k \rangle. \tag{1.7}$$

H is directly diagonalized, yielding

$$H \phi_k = \lambda_k \phi_k, \tag{1.8}$$

with estimated eigenpairs $(\lambda_k^{-1}, V\phi_k)$ for A, where V is the rectangular matrix whose j^{th} column is v_j. A shift s can, as usual, be used to accelerate convergence of the Arnoldi method to a desired eigenvalue. In this case, we solve

$$(A - sI) u_{k+1} = u_k. \tag{1.9}$$

Solving the linear systems (1.6) or (1.9) is by far the most time-consuming part of the algorithm. This means that it is far more difficult to find the smallest eigenvalues of A than the largest ones, since acting with A is usually far easier than acting with its inverse. The purpose of this paper is to present a method for quickly formulating and solving the linear systems (1.6) or (1.9), assuming that we have a time-stepping code for integrating the time-dependent equation (1.1).

A related scheme has been used to compute steady states via Newton's method [2–6]. This scheme has been proposed and used as a method for calculating eigenvalues in [7–10]. Here we provide a study of its convergence.

2 Method

2.1 Laplacian preconditioning

Our method for solving (1.6) is based on the BiCGSTAB variant of the conjugate gradient method [11]. Since A results from the spatial discretization of a partial differential equation, its size may be quite large. Denoting by M the number of points or modes necessary