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New Finite Difference Methods Based on IIM for Inextensible Interfaces in Incompressible Flows

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Abstract. In this paper, new finite difference methods based on the augmented immersed interface method (IIM) are proposed for simulating an inextensible moving interface in an incompressible two-dimensional flow. The mathematical models arise from studying the deformation of red blood cells in mathematical biology. The governing equations are incompressible Stokes or Navier-Stokes equations with an unknown surface tension, which should be determined in such a way that the surface divergence of the velocity is zero along the interface. Thus, the area enclosed by the interface and the total length of the interface should be conserved during the evolution process. Because of the nonlinear and coupling nature of the problem, direct discretization by applying the immersed boundary or immersed interface method yields complex nonlinear systems to be solved. In our new methods, we treat the unknown surface tension as an augmented variable so that the augmented IIM can be applied. Since finding the unknown surface tension is essentially an inverse problem that is sensitive to perturbations, our regularization strategy is to introduce a controlled tangential force along the interface, which leads to a least squares problem. For Stokes equations, the forward solver at one time level involves solving three Poisson equations with an interface. For Navier-Stokes equations, we propose a modified projection method that can enforce the pressure jump condition corresponding directly to the unknown surface tension. Several numerical experiments show good agreement with other results in the literature and reveal some interesting phenomena.

AMS subject classifications: 65M06, 65M12, 76T05

Key words: Inextensible interface, incompressible flow, Stokes equations, Navier-Stokes equations, immersed interface method, inverse problem, regularization, augmented immersed interface method.

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1. Introduction

In this paper, we develop some new finite difference methods based on the augmented immersed interface method (cf. for example [2, 7, 10, 12]) for simulating an inextensible moving interface in an incompressible shear flow. The problem involves finding an unknown *surface tension* $\sigma(s,t)$ such that the surface divergence of the velocity is zero along the interface. Since the fluid is incompressible and the interface is inextensible, both the area enclosed by the interface and the total length of the interface should be conserved. The mathematical model has been used to describe the deformation of erythrocytes, also called red blood cells in the field of bio-rheology (cf. [4, 13–19] and the references therein for the bio-mathematical applications and other related information).

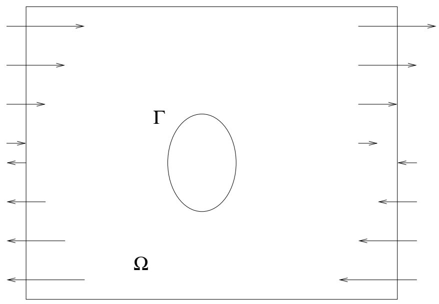

Figure 1: A diagram of an inextensible interface in a shear flow.

The fluid equations can be formulated by either the Stokes equations (the inertial term is neglected)

$$\nabla p = \mu \Delta u + \mathbf{F}(\mathbf{x},t), \qquad \mathbf{x} \in \Omega, \tag{1.1}$$

or the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mu \Delta u + \mathbf{F}(\mathbf{x},t), \qquad \mathbf{x} \in \Omega, \tag{1.2}$$

with the fluid incompressibility constraint

$$\nabla \cdot \mathbf{u} = 0. \tag{1.3}$$

Here, we assume that the interface motion is under a shear flow $\mathbf{u} = \dot{\gamma}\, y\, \mathbf{e}_1$ along the boundary of a finite domain Ω as illustrated in Fig. 1, where $\dot{\gamma}$ is the shear rate (a fixed number). The force term $\mathbf{F}(\mathbf{x},t)$ has the form

$$\mathbf{F}(\mathbf{x},t) = \int_{\Gamma(t)} \left\{ \frac{\partial}{\partial s} \Big(\sigma(s,t)\, \boldsymbol{\tau}(s,t) \Big) + f_b\, \mathbf{n} + g(s,t)\boldsymbol{\tau}(s,t) \right\} \delta(\mathbf{x} - \mathbf{X}(s,t))\, ds, \tag{1.4}$$

where $\mathbf{X}(s,t)$ is a parametric representation of the moving interface Γ, and $\{\mathbf{n}, \boldsymbol{\tau}\}$ are the unit normal and tangential directions of the moving interface, respectively. The bending