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Minimisation and Parameter Estimation in Image Restoration Variational Models with ℓ_1-Constraints

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Abstract. Minimisation of the total variation regularisation for linear operators under ℓ_1-constraints applied to image restoration is considered, and relationships between the Lagrange multiplier for a constrained model and the regularisation parameter in an unconstrained model are established. A constrained ℓ_1-problem reformulated as a separable convex problem is solved by the alternating direction method of multipliers that includes two sequences, converging to a restored image and the “optimal" regularisation parameter. This allows blurry images to be recovered, with a simultaneous estimation of the regularisation parameter. The noise level parameter is estimated, and numerical experiments illustrate the efficiency of the new approach.

AMS subject classifications: 68U10, 65J22, 65K10, 90C25

Key words: Parameter selection, ℓ_1-constraints, alternating direction method of multipliers, impulsive noise, image processing.

1. Introduction

The restoration of images is a challenging problem, and here images $\bar{x} \in \mathbb{R}^n$ corrupted by impulse noise and blurring effects are discussed. Blurring issues are almost unavoidable in contemporary imaging systems, and corruption by impulse noise emerges from bit errors in transmission, wrong pixels and faulty memory locations in hardware [1, 6, 12, 23]. In the model, an observed image f is represented by the equation

$$f = N(K\bar{x}), \tag{1.1}$$

where N and K denote the impulse noise and blurring effect, respectively. In applications, there are two main impulse noise sources — viz. salt-and-pepper and random-valued impulse noise [9]. Image restoration problems are usually ill conditioned and the direct solution of the system (1.1) rarely produces satisfactory results. To address this problem, one can use a regularisation procedure — e.g. the total variational (TV) regularisation [35], a

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wavelet transform [8], or a nonlocal regularisation [21]. The corresponding unconstrained model for the image deblurring can be written in the form

$$\min_{x\in\mathbb{R}^n}\{\|Dx\|+\lambda\Phi_{\text{fit}}(x,f)\}\,, \tag{1.2}$$

where D is a linear operator and $\Phi_{\text{fit}}(x,f)$ is the data-fitting term. In particular, if D is the discrete gradient operator ∇, then this model (1.2) represents the TV regularisation where the parameter λ balances the data fidelity with the regularity. The constrained counterpart of the model (1.2) is

$$\begin{array}{cl} \min_{x\in\mathbb{R}^n} & \|Dx\| \\ s.t. & \Phi_{\text{fit}}(x,f)\le\tau \end{array} \tag{1.3}$$

with τ representing the noise level, and constrained models have been also used in image recovery [32, 41, 45]. For the impulse noise, one sets $\Phi_{\text{fit}}(x,f)=\|Kx-f\|_1$ in either of (1.2) or (1.3), which are respectively called unconstrained and constrained ℓ_1-models.

Numerical methods have been used to solve the unconstrained model (1.2) under the TV regularisation [17, 40, 45], a wavelet regulariser [14], and a nonlocal regularisation [15]. The associated minimiser $\hat{x}=\hat{x}(\lambda)$, and hence the recovered image, depends on the choice of λ. Usually, the parameter λ is determined manually by a trial-and-error method, but several techniques have been developed to detect the best parameter λ automatically — e.g. the L-curve criterion [24], the generalised cross-validation (GCV) [19,20], the normalised cumulative or residual periodogram approach [25,36], variational Bayes approaches [2, 3, 34], and Morozov's discrepancy principle (MDP) [31]. On the other hand, constrained ℓ_1-models have only recently been considered, and if *a priori* information about noise is available then the constrained model is more attractive [41, 47].

To solve unconstrained ℓ_1-models, various algorithms have been developed recently. In particular, a fast TV deblurring algorithm (FTVd) combines the variable splitting and quadratic penalised technique [40]. Each subproblem can be solved by either shrinkage or the fast Fourier transform (FFT), so the FTVd method performs much better than many other methods [17, 42]. An incremental version of the FTVd approach involving the alternating direction method of multipliers (ADMM) can be used to solve an unconstrained TV-ℓ_1-model. Wu *et al.* [44] developed the inexact augmented Lagrangian method equivalent to the ADMM. All of these approaches assume that a suitable regularisation parameter in the unconstrained model is known — and it can be determined manually by a trial-and-error method, but this procedure is often very slow, so one may prefer to restore images corrupted by impulse noise via the constrained TV-ℓ_1-model. Weiss *et al.* [45] used Nesterov's first-order scheme to do so, and this approach was improved by Ng *et al.* [33], but it requires inner iterations to describe projections onto an ℓ_1-ball.

The parameter λ in the unconstrained model (1.2) can also be determined by some classical methods — e.g. if the regularisation term has a quadratic form the GCV evaluation formula may be employed. However, this formula cannot be used directly in the seminorm-based model (1.2) due to the non-linearity of the seminorms $\|D\cdot\|_2$. The regularisation parameter can be estimated by quadratic approximation of the seminorm term [30], or by