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A Nontrivial Solution to a Stochastic Matrix Equation

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Abstract. If A is a nonsingular matrix such that its inverse is a stochastic matrix, the classic Brouwer fixed point theorem implies that the matrix equation $AXA = XAX$ has a nontrivial solution. An explicit expression of this nontrivial solution is found via the mean ergodic theorem, and fixed point iteration is considered to find a nontrivial solution.

Key words: Matrix equation, Brouwer's fixed point theorem.

1. Introduction

The matrix equation

$$ABA = BAB, \tag{1.1}$$

where both A and B are square matrices of the same size, is closely related to the parameter-independent Yang-Baxter equation (independently introduced by C. N. Yang in 1968 and Rodney Baxter in 1971 in statistical mechanics) and the theory of braid groups. The Yang-Baxter equation and braid groups, together with knot theory, have been extensively studied by physicists and mathematicians in the past decades — cf. Ref. [4] for more details on the Yang-Baxter equation and related topics. However, to our knowledge even the relevant simple matrix equation (1.1) has not been seriously explored in matrix theory.

Finding all pairs of matrices (A,B) that satisfy Eq. (1.1) is no trivial task. Thus given one of the two $n \times n$ matrices A and B (say A), finding B to satisfy the matrix equation (1.1) is equivalent to solving a system of n^2 quadratic equations. The solution to a system

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of polynomial equations is a major topic in algebraic geometry, and it is not easy to find all of the solutions of the system even with 3×3 matrices.

In this paper, we are interested in finding all solutions of the following matrix equation

$$AXA = XAX \tag{1.2}$$

for a given invertible matrix A such that A^{-1} is a stochastic matrix. A matrix is called *stochastic* if it is a nonnegative matrix such that each of its row sums equals 1.

Most matrix equations that have been studied are linear in the unknown matrix, and there are many established tools and methods to solve linear matrix equations. For example, the vector space structure of the solution set to the equation $AX = XA$ (for all matrices that commute with A) is determined by the Jordan form structure of the matrix A [1]. However, the matrix equation (1.2) is nonlinear, and there is no general algebraic theory to assist in finding its solution. Eq. (1.2) obviously has two trivial solutions, the zero solution and the solution of $X = A$. In this paper, we show that Eq. (1.2) has nontrivial solutions when A^{-1} is a stochastic matrix, one of which is a stochastic solution. We may also regard Eq. (1.2) as a stochastic matrix equation, for which we want to find a stochastic solution. For another example of a stochastic matrix equation, see Refs. [5,6,8] and other references therein.

In the next section, we use the classical Brouwer fixed point theorem to establish the existence of a nontrivial solution to Eq. (1.2). This famous theorem says that if f is a continuous mapping from a compact convex set D into itself, then f has a fixed point — i.e. $f(x^*) = x^*$, where $x^* \in D$. An explicit form of this nontrivial solution is found in Section 3, by invoking the mean ergodic theorem for stochastic matrices. Another explicit nontrivial solution is also obtained in Section 4, due to the special structure of the matrix equation. A discussion on computing nontrivial solutions is presented in Section 5, and we summarize our results in Section 6.

2. The Existence of a Nontrivial Solution

The following Theorem proves the existence of a solution to Eq. (1.2).

Theorem 2.1. *Suppose $A \in R^{n\times n}$ is invertible such that its inverse A^{-1} is a stochastic matrix. Then the equation* (1.2) *has a solution $X^* = Z^* A^{-1}$ where Z^* is a stochastic matrix, and hence X^* is a stochastic matrix.*

Proof. Write the equation (1.2) as

$$XA = A^{-1}XAXAA^{-1} .$$

Let $Z = XA$. Then the above equation is the following fixed point equation for Z:

$$Z = A^{-1}Z^2A^{-1} .$$