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Newton-Shamanskii Method for a Quadratic Matrix Equation Arising in Quasi-Birth-Death Problems

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Abstract. In order to determine the stationary distribution for discrete time quasi-birth-death Markov chains, it is necessary to find the minimal nonnegative solution of a quadratic matrix equation. The Newton-Shamanskii method is applied to solve this equation, and the sequence of matrices produced is monotonically increasing and converges to its minimal nonnegative solution. Numerical results illustrate the effectiveness of this procedure.

AMS subject classifications: 65F30, 65H10

Key words: Quadratic matrix equation, quasi-birth-death problems, Newton-Shamanskii method, minimal nonnegative solution.

1. Introduction

Some necessary notation for this article is as follows. For any matrix $B = [b_{ij}] \in \mathbb{R}^{n\times n}$, $B \geq 0$ ($B > 0$) if $b_{ij} \geq 0$ ($b_{ij} > 0$) for all i, j; for any matrices $A, B \in \mathbb{R}^{n\times n}$, $A \geq B$ ($A > B$) if $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$) for all i, j; for any vectors $x, y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) if $x_i \geq y_i$ ($x_i > y_i$) for all $i = 1, \cdots, n$; the vector with all entries one is denoted by e — i.e. $e = (1, 1, \cdots, 1)^T$; and the identity matrix is denoted by I. The quadratic matrix equation (QME)

$$\mathcal{Q}(X) = AX^2 + BX + C = 0 \tag{1.1}$$

is considered, where $A, B, C, X \in \mathbb{R}^{n\times n}$, $A, B + I, C \geq 0$, $A + B + I + C$ is irreducible and $(A + B + C)e = e$. This quadratic matrix equation arises in quasi-birth-death processes (QBD) [4], and its element-wise minimal nonnegative solution S is of particular interest. The rate ρ of a QBD Markov chain is defined by

$$\rho = p^T(B + I + 2A)e\,, \tag{1.2}$$

where p is the stationary probability vector of the stochastic matrix $A + B + I + C$ — i.e. $p^T(A + B + I + C) = p^T$ and $p^T e = 1$ (cf. the monograph [4] for further details). A QBD is

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said to be positive recurrent if $\rho < 1$, null recurrent if $\rho = 1$ and transient if $\rho > 1$ — and throughout this article the QBD is assumed to be positive recurrent.

There have been several numerical methods proposed to solve the QME (1.1). Some linearly convergent fixed point iterations are summarised and analysed in Ref. [1] and references therein. Latouche [2] showed the application of Newton's algorithm is well defined, and that the matrix sequence is monotonically increasing and quadratically convergent. An invariant subspace method approximates the minimal nonnegative solution S quadratically, through approximating the left invariant subspace of a block companion matrix [4,9]. Latouche & Ramaswami [5] proposed a logarithmic reduction algorithm generating sequences of approximations that converge quadratically to S, based on a divide-and conquer strategy. Bini *et al.* [6–10] proposed a quadratically convergent and numerically stable algorithm for the computation of S based on a functional representation of cyclic reduction, which applies to general M/G/1-type Markov chains [16] and generalises the method of Ref. [5]. Poloni [12] studied several quadratic vector and matrix equations with nonnegativity hypotheses in a unified fashion, giving further insight into the equations. The Newton-Shamanskii method has been proposed for other equations — e.g. the vector equation arising in transport theory [13], the algebraic Riccati equation with four coefficient matrices forming a nonsingular M-matrix or an irreducible singular M-matrix [14], and the vector equation arising in Markovian binary trees [15].

In this article, the Newton-Shamanskii method is applied to the QME (1.1). Newton's method is recalled and the Newton-Shamanskii iterative procedure is presented in Section 2. Then in Section 3 it is shown that, starting with a suitable initial guess, the sequence of iterative matrices generated by the Newton-Shamanskii method is monotonically increasing and converges to the minimal nonnegative solution of the QME (1.1). Numerical results in Section 4 show that the Newton-Shamanskii method can be more efficient than the Newton method. Final conclusions are presented in Section 5.

2. Newton-Shamanskii Method

The function $\mathcal{Q}$ in the QME (1.1) is a mapping from $\mathbb{R}^{n\times n}$ into itself, and the Fréchet derivative of $\mathcal{Q}$ at X is a linear map $\mathcal{Q}_X^{'} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by

$$\mathcal{Q}_X^{'}(Z) = AZX + AXZ + BZ. \tag{2.1}$$

The second derivative $\mathcal{Q}_X^{''} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ at X is given by

$$\mathcal{Q}_X^{''}(Z_1, Z_2) = AZ_1Z_2 + AZ_2Z_1. \tag{2.2}$$

For given X_0, the Newton sequence for the solution of $\mathcal{Q}(X) = 0$ is

$$X_{k+1} = X_k - (\mathcal{Q}_{X_k}^{'})^{-1}\mathcal{Q}(X_k), \quad k = 0, 1, 2, \cdots, \tag{2.3}$$

provided that $\mathcal{Q}_{X_k}^{'}$ is invertible for all k. From Eq. (2.1), the Newton iteration (2.3) is equivalent to

$$\begin{cases} AZX_k + (AX_k + B)Z = -\mathcal{Q}(X_k)\,, \\ X_{k+1} = X_k + Z\,, \quad k = 0, 1, 2, \cdots \end{cases} \tag{2.4}$$