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Generalised (2+1)-dimensional Super MKdV Hierarchy for Integrable Systems in Soliton Theory

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Abstract. Much attention has been given to constructing Lie and Lie superalgebra for integrable systems in soliton theory, which often have significant scientific applications. However, this has mostly been confined to (1+1)-dimensional integrable systems, and there has been very little work on (2+1)-dimensional integrable systems. In this article, we construct a class of generalised Lie superalgebra that differs from more common Lie superalgebra to generate a (2+1)-dimensional super modified Korteweg-de Vries (mKdV) hierarchy, via a generalised Tu scheme based on the Lax pair method where the Hamiltonian structure derives from a generalised supertrace identity. We also obtain some solutions of the (2+1)-dimensional mKdV equation using the G'/G^2 method.

AMS subject classifications: 35Q51, 37K05, 37K10

Key words: Soliton theory, generalised Lie superalgebra, (2+1)-dimensional super mKdV hierarchy, supertrace identity, G'/G^2 method.

1. Introduction

In soliton theory, new physically justified integrable and super integrable systems are often sought, and different methods for generating integrable and super integrable systems have previously been proposed — e.g. Refs. [1–4]. An efficient approach due to Tu [5] generates integrable systems as well as Hamiltonian structures based on the Lax pair method, and following Ma [6] we call this approach the Tu scheme. Some researchers have applied the Tu method to obtain (1+1)-dimensional integrable and super integrable systems [7–12]. With the development of solitary wave theory, attention has turned to more applicable (2+1)-dimensional integrable systems. Tu himself [13] obtained the Kadomtsev-Petviashvili (KP) and Davey-Stewartson(DS) equations, and recently Zhang *et al.* [14, 15] discussed several other (2+1)-dimensional equations. Various methods have been used to solve the (1+1)-dimensional equations [16–21], and solutions of (2+1)-dimensional and even (3+1)-dimensional nonlinear partial differential equations have also been found [22, 23].

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In this article, we use the generalised Tu scheme to generate a (2+1)-dimensional super mKdV hierarchy and obtain some solutions. The main steps of the Tu scheme are as follows.

(1) Define an appropriate matrix $B = \{(b_{ij})_{n\times n}\} = \{f_1, f_2, \cdots, f_p\}$, to express the loop Lie superalgebra as $\widetilde{B} = \{B\lambda^n\} = \{f_1(n), f_2(n), \cdots, f_p(n)\} = \{f_1\lambda^n f_2\lambda^n, f_3\lambda^n, \cdots f_p\lambda^n\}$.
(2) Construct the isospectral problem $\varphi_x = UM\varphi$, $\varphi_t = VM\varphi$ involving two matrices U and V, where $U = u_1 f_1(n) + u_2 f_2(n) + u_3 f_3(n) + \cdots + u_p f_p(n)$ and $u_1, u_2, u_3, \cdots, u_p$ are functions of x, y and t, ψ is a vector function, and $M \in B$.
(3) Solve the static zero curvature equation $V_x = [U, V] = UMV - VMU$ where $V = \sum V_m \lambda^{-m}$, to get recursive equations.
(4) Establish $-V^n_{+x} + [U, V^{(n)}_+] \in C_1 f_1 + C_2 f_2 + C_3 f_3 + \cdots + C_p f_p$; or if not, construct a correction term $\Delta_n \in \widetilde{B}$ such that $V^{(n)} = V^{(n)}_+ + \Delta_n$ and $-V^{(n)}_x + [U, V^{(n)}] \in C_1 f_1 + C_2 f_2 + C_3 f_3 + \cdots + C_p f_p$.
(5) Solve the equation $U_t - V^n_x + [U, V^{(n)}] = 0$ to obtain $U_{t_n} = J g^{(n)}$, where $g^{(n)}$ satisfies $\langle V, \partial U/\partial u_i \rangle = \sum g^{(n)} \lambda^{-n}$ and J is a Hamiltonian operator; and then calculate $g^{(n)} = L g^{(n-1)}$ to determine the recurrence operator L.
(6) Use the (2+1)-dimensional supertrace identity to find the Hamiltonian structure.

Further, in proceeding to the (2+1)-dimensional super mKdV hierarchy and its Hamiltonian structure of interest here, some further operators and ideas are needed as follows.
(1) We constitute an associative algebra $\mathscr{A}[\xi]$, which includes all pseudo-differential operators $\sum_{i=-\infty}^{N} a_i \xi^i$, $a_i \in \mathscr{A}$, where ξ represent an operator defined by

$$\xi f = f\xi + \partial f, \quad f \in \mathscr{A}\,; \tag{1.1}$$

and on the basis of Eq. (1.1) infer that

$$\xi^n f = \sum_{n\geq 0} \binom{n}{i} (\partial^i f) \xi^{n-i},$$
$$f\xi^n = \sum_{n\geq 0} (-1) \binom{n}{i} \xi^{n-i} (\partial^i f), \quad n \in Z\,,$$

and introduce a residue operator $R : \mathscr{A}[\xi] \longrightarrow \mathscr{A}, R(\sum a_i \xi^i) = a_{-1}$.
(2) Adjusting the derivation $\partial : \partial = \partial_y$, we obtain the algebra $\mathscr{A}[\xi]$ involving the operator ξ such that

$$\xi f = f\xi + \partial_y f, \quad f \in \mathscr{A}, \tag{1.2}$$

and expand the operators ∂_x and ∂_y from $\mathscr{A}$ to $\mathscr{A}[\xi]$:

$$\partial_x \left(\sum a_i \xi^i \right) = \sum (\partial_x a_i) \xi^i\,,$$
$$\partial_y \left(\sum a_i \xi^i \right) = \sum \left(\partial_y a_i \right) \xi^i\,.$$