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A Numerical Comparison of Finite Difference and Finite Element Methods for a Stochastic Differential Equation with Polynomial Chaos

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Abstract. A numerical comparison of finite difference (FD) and finite element (FE) methods for a stochastic ordinary differential equation is made. The stochastic ordinary differential equation is turned into a set of ordinary differential equations by applying polynomial chaos, and the FD and FE methods are then implemented. The resulting numerical solutions are all non-negative. When orthogonal polynomials are used for either continuous or discrete processes, numerical experiments also show that the FE method is more accurate and efficient than the FD method.

AMS subject classifications: 60H10, 60H35, 65L12, 74S05

Key words: Stochastic differential equation, polynomial chaos, finite difference method, finite element method, non-negative solution.

1. Introduction

We consider numerical solutions of the stochastic ordinary differential equation

$$\begin{aligned} &\frac{\mathrm{d}y(t;\omega)}{\mathrm{d}t} = -k(\omega)y(t;\omega) + g(t), \quad t \in (0,T]\,, \\ &y(0;\omega) = \hat{y}, \end{aligned} \tag{1.1}$$

where the decay rate coefficient $k(\omega)$ is a random variable, $\omega \in \Omega$ where Ω is all possible event space, and $\hat{y}$ is the initial value. For given values of k we have the exact solution $y(t) = c_0 e^{-kt} + \int_0^t e^{k(s-t)} g(s)\,\mathrm{d}s$, where the constant c_0 is determined from the initial condition. On the other hand, for a random input $k(\omega)$ the method of polynomial chaos can be invoked, where both $k(\omega)$ and $y(t;\omega)$ may be expanded in series to get the solution.

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Wiener [11] introduced the concept of chaos, when representing a Gaussian process as an expansion in Hermite polynomials. Ghanem & Spanos [3–5] combined the Hermite expansion with the finite element (FE) method to model the uncertainty encountered in various problems of solid mechanics. According to the Cameron-Martin theorem [1], a series in Hermite polynomials for any stochastic process with finite second-order moment converges. However, while the convergence is fast (exponential) for Gaussian random processes, for other stochastic processes the convergence speed of Hermite polynomial chaos it can be very slow indeed. To overcome this drawback, Tang [16] introduced an Hermite spectral method for Gaussian-type functions, and Tang & Zhou [17] considered discrete least square projections in unbounded domains with random evaluations and the application of their approach to quantification of parameter uncertainty. Further, the convergence speed is improved by using a scaling factor or polynomials other than Hermite.

Later, Ogura [8] used Charlier polynomial chaos in simulating the Poisson process, and Xiu & Karniadakis presented a generalised polynomial chaos method based on Askey polynomials [12–15]. In particular, stochastic processes with optimum trial bases from the Askey family of orthogonal polynomials reduced the dimensionality of the system, leading to exponential convergence for the error [12]. These methods usually require finite truncation and Galerkin projection. Furthermore, each member in the family of Askey polynomial chaos has distinctive mutually orthogonal weight functions that are typically consistent with the probability density function for the stochastic process, whether continuous or discrete. Thus Hermite polynomials correspond to the Gaussian distribution, Laguerre polynomials to the Gamma distribution, Charlier polynomials to the Poisson distribution and Krawtchouk polynomials correspond to the Binomial distribution, when representing different stochastic processes with different polynomial chaos in order to achieve the optimal approximation rate.

Following Xiu & Karniadakis [12], in this article we use FD and FE methods to solve the resulting ordinary differential equations numerically, and consider the effectiveness of certain time-discretization schemes. Moreover, through special handling of the orthogonal polynomials we can guarantee that all of the numerical solutions of the resulting ordinary differential equations are non-negative, and find that the FE method proves better results than either the FD method or the first-order and second-order Runge-Kutta schemes envisaged in Ref. [12]. The application of polynomial chaos is introduced in Section 2. In Section 3, we present our FD and FE methods for solving the random input inhomogeneous ordinary differential equations, and find non-negative solutions together with some error estimates. We present our numerical results for various distributions in Section 4, and our brief concluding remarks are in Section 5.

2. Application of Wiener-Askey Polynomial Chaos

We first briefly introduce some properties of the orthogonal polynomials. Since the random variable obeys different distributions associated with different orthogonal polynomials chaos, the orthogonal polynomials $\{\Phi_n,\ n = 0, 1, \cdots\}$ satisfy the following relationships.