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A High Order Well-Balanced Finite Volume WENO Scheme for a Blood Flow Model in Arteries

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Abstract. The numerical simulations for the blood flow in arteries by high order accurate schemes have a wide range of applications in medical engineering. The blood flow model admits the steady state solutions, in which the flux gradient is non-zero and is exactly balanced by the source term. In this paper, we present a high order finite volume weighted essentially non-oscillatory (WENO) scheme, which preserves the steady state solutions and maintains genuine high order accuracy for general solutions. The well-balanced property is obtained by a novel source term reformulation and discretisation, combined with well-balanced numerical fluxes. Extensive numerical experiments are carried out to verify well-balanced property, high order accuracy, as well as good resolution for smooth and discontinuous solutions.

AMS subject classifications: 74S10

Key words: Blood flow model, finite volume scheme, WENO scheme, well-balanced property, high order accuracy, source term.

1. Introduction

The numerical simulations for the blood flow in arteries by high order accurate schemes have a wide range of applications in medical engineering [5,8]. The blood flow in arteries can be described by the following one-dimensional blood flow model:

$$\begin{cases} A_t + Q_x = 0, \\ Q_t + \left(\frac{Q^2}{A} + \frac{K}{3\rho\sqrt{\pi}}A^{\frac{3}{2}}\right)_x = \frac{KA}{2\rho\sqrt{\pi}\sqrt{A_0}}(A_0)_x, \end{cases} \tag{1.1}$$

where A is the cross-sectional area (i.e., $A = \pi R^2$ with R being the radius of the vessel), $Q = Au$ represents the discharge, u means the flow velocity, and ρ stands for the blood density. In addition, K denotes the arterial stiffness, and A_0 is the cross section at rest (i.e., $A_0 = \pi R_0^2$ with R_0 being the radius of the vessel, which may be variable in the case of

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aneurism, stenosis or taper). In general, the original governing equations (1.1) take the following form

$$U_t + F(U)_x = S(A, A_0), \tag{1.2}$$

where U is the solution vector with the corresponding flux $F(U)$, and $S(A, A_0)$ stands for the source term.

The blood flow model (1.1) belongs to the class of hyperbolic equations with source term (also referred as hyperbolic balance laws). The important property of this system is that it usually admits non-trivial steady state solutions, also called mechanical equilibrium:

$$u = 0, \quad A = A_0, \tag{1.3}$$

in which the source term is exactly balanced by the flux gradient. There is a main challenge in the numerical simulation of such balance laws, i.e., the standard numerical schemes may not satisfy the discrete version of this balance exactly at (or near) the steady state, and may introduce spurious oscillations, when the mesh is not extremely refined. However, the mesh refinement approach is not practical for the high dimensional problems. Therefore, well-balanced schemes [6] are specially designed to preserve exactly these steady state solutions up to the machine accuracy for the purpose of saving the computational cost. In addition, the well-balanced schemes can capture small perturbations on relatively coarse meshes [17]. We can refer to [11] for more information about the well-balanced schemes.

In the last few years, there have been many interesting attempts proposed in the literature to derive well-balanced schemes for the blood flow model. For instance, based on the conservative governing equations [1,2,15] and the hydrostatic reconstruction, Delestre and Lagrée [3] presented a well-balanced finite volume scheme for the blood flow model. Recently, Müller et al. [10] constructed a well-balanced high order finite volume scheme for the blood flow in elastic vessels with varying mechanical properties. And Murillo et al. [9] designed an energy-balanced approximate solver for the blood flow model with upwind discretisation for the source term. More recently, Wang et al. [16] have presented a high order well-balanced finite difference weighted essentially non-oscillatory (WENO) scheme.

During the past few decades, high order finite volume WENO schemes have gained great attentions in solving hyperbolic conservation laws [12]. Several advantages of the WENO schemes, including its accuracy and essentially non-oscillatory property, make it useful for a wide range of applications.

The main objective of this paper is to present a high order accurate finite volume well-balanced WENO scheme for the blood flow model in arteries. To achieve the well-balanced property, we firstly reformulate the source terms in an equivalent form by using the steady state (1.3). Then, we apply well-balanced numerical fluxes accordingly.

In Section 2, we present a novel high order well-balanced finite volume WENO scheme. Section 3 contains extensive numerical results to demonstrate the behavior of our well-balanced WENO scheme for the blood flow model, verifying high order accuracy, the well-balanced property, and good resolution for smooth and discontinuous solutions. Some conclusions are given in Section 4.