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Solution of a Nonlinear Eigenvalue Problem Using Signed Singular Values

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Abstract. We propose a robust numerical algorithm for solving the nonlinear eigenvalue problem $A(\lambda)\mathbf{x} = \mathbf{0}$. Our algorithm is based on the idea of finding the value of λ for which $A(\lambda)$ is singular by computing the smallest eigenvalue or singular value of $A(\lambda)$ viewed as a constant matrix. To further enhance computational efficiency, we introduce and use the concept of signed singular value. Our method is applicable when $A(\lambda)$ is large and nonsymmetric and has strong nonlinearity. Numerical experiments on a nonlinear eigenvalue problem arising in the computation of scaling exponent in turbulent flow show robustness and effectiveness of our method.

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1. Introduction

Let $A(\lambda) \in \mathbf{R}^{n\times n}$ be a matrix whose elements depend on a scalar parameter λ. In the *nonlinear eigenvalue problem*, we seek a value of λ for which there exists a nonzero vector $\mathbf{x} \in \mathbf{R}^n$ such that

$$A(\lambda)\mathbf{x} = \mathbf{0} \tag{1.1}$$

holds. In this paper, we call λ and $\mathbf{x}$ the *nonlinear eigenvalue* and *nonlinear eigenvector*, respectively. Eq. (1.1) includes many types of eigenvalue problems as a special case. When $A(\lambda) = A - \lambda B$, we have a generalized eigenvalue problem. $A(\lambda) = \lambda^2 M + \lambda C + K$ leads to a quadratic eigenvalue problem. When $A(\lambda) = (e^{\lambda} - 1)A_1 + \lambda^2 A_2 + A_3$, we have a general nonlinear eigenvalue problem with exponential dependence on λ.

Nonlinear eigenvalue problems arise in a variety of applications. In structural mechanics, a decay system is described by a quadratic eigenvalue problem [16]. In electronic

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structure calculation, the augmented plane wave (APW) method [9] gives rise to a nonlinear eigenvalue problem $H(\lambda)\mathbf{x} = \lambda\mathbf{x}$, where $H(\lambda)$ is the APW Hamiltonian. In theoretical fluid dynamics, computation of the scaling exponent in turbulent flow is formulated as a nonlinear eigenvalue problem [8]. Among these problems, the last one is hard to solve numerically, because the coefficient matrix $A(\lambda)$ is large and nonsymmetric, and dependence on λ is strongly nonlinear. Also, the elements of $A(\lambda)$ are not given explicitly, but are given by a computer program. The cost of computing $A(\lambda)$ itself for a given λ is very large.

Several numerical algorithms have been proposed so far to solve nonlinear eigenvalue problems. There are algorithms based on multivariate Newton's method [10, 11], nonlinear extensions of the Arnoldi [17] and Jacobi-Davidson [4, 18] algorithms, and algorithms based on complex contour integral [2, 3, 20]. While these methods are efficient for certain types of problems, they are not suited to a problem like the one arising in the computation of the scaling exponent in turbulent flow. Multivariate Newton's method and nonlinear extensions of Arnoldi and Jacobi-Davidson methods are based on linear approximation, and require either that the target problem is only weakly nonlinear or sufficiently good initial approximations of the eigenvalue and the eigenvector are provided. Algorithms based on complex contour integral can be applied even when the problem is strongly nonlinear. But they require evaluation of $A(\lambda)$ for complex values of λ, even when the nonlinear eigenvalue to be computed is real.

In this paper, we propose a robust numerical method for the nonlinear eigenvalue problem. In our method, we compute the nonlinear eigenvalue by seeking the value of λ for which $A(\lambda)$ is singular. As a measure of singularity, we use the *linear eigenvalue* of $A(\lambda)$, which is simply an eigenvalue of $A(\lambda)$ viewed as a constant matrix. Thus the problem is reduced to finding the zero of the linear eigenvalue as a function of λ. The latter problem can be solved stably even if the original problem has strong nonlinearity, because it is a one-dimensional root finding problem. To further improve the computational efficiency, we introduce the notion of the *signed singular value* of $A(\lambda)$ and propose to use it instead of the linear eigenvalue. Numerical experiments show that these methods can solve the nonlinear eigenvalue problem such as the one arising in the computation of the scaling exponent in turbulent flow stably and efficiently.

The rest of this paper is organized as follows. In Section 2, we explain our target nonlinear eigenvalue problem in more detail and show the difficulties with the existing algorithms. To overcome the difficulties, we present two algorithms, namely, the one based on the linear eigenvalue and the one based on the signed singular value in Section 3. The effectiveness of these methods is examined through numerical experiments in Section 4. Section 5 is devoted to conclusions.

2. The Target Problem and Existing Algorithms

2.1. Target problems

In this paper, we consider a nonlinear eigenvalue problem that has the following characteristics.