INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 15, Number 1-2, Pages 1–25

POSITIVITY-PRESERVING HIGH-ORDER SCHEMES FOR CONSERVATION LAWS ON ARBITRARILY DISTRIBUTED POINT CLOUDS WITH A SIMPLE WENO LIMITER

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Abstract. This is an extension of our earlier work [9] in which a high order stable method was constructed for solving hyperbolic conservation laws on arbitrarily distributed point clouds. An algorithm of building a suitable polygonal mesh based on the random points was given and the traditional discontinuous Galerkin (DG) method was adopted on the constructed polygonal mesh. Numerical results in [9] show that the current scheme will generate spurious numerical oscillations when dealing with solutions containing strong shocks. In this paper, we adapt a simple weighted essentially non-oscillatory (WENO) limiter, originally designed for DG schemes on two-dimensional unstructured triangular meshes [27], to our high order method on polygonal meshes. The objective of this simple WENO limiter is to simultaneously maintain uniform high order accuracy of the original method in smooth regions and control spurious numerical oscillations near discontinuities. The WENO limiter we adopt is particularly simple to implement and will not harm the conservativeness and compactness of the original method. Moreover, we also extend the maximum-principle-satisfying limiter for the scalar case and the positivity-preserving limiter for the Euler system to our method. Numerical results for both scalar equations and Euler systems of compressible gas dynamics are provided to illustrate the good behavior of these limiters.

Key words. WENO limiter, positivity-preserving, arbitrarily distributed point cloud, conservation laws, high order.

1. Introduction

In this paper, we are interested in solving the following two-dimensional hyperbolic conservation law

$$\begin{cases} u_t + \nabla \cdot \mathbf{F}(u) = 0, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \end{cases} \tag{1}$$

in the computational domain $\Omega \subset \mathbb{R}^2$, where $\mathbf{F} = (f(u), g(u))$ is the flux vector, $\mathbf{x} = (x, y) \in \Omega$ and $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. Here, u, f and g can be either scalars or vectors. We assume that an arbitrarily distributed point cloud, namely a finite set of isolated, unstructured points $\{\mathbf{x}_i\}_{i=1}^N$, together with the values of the initial condition at these points $\{u_0(\mathbf{x}_i),\, i = 1, \cdots, N\}$, is given, and we seek an algorithm to obtain the point values of the numerical solution in this point cloud for later time. One possible scenario for such a set-up could be that the point clouds are locations of the observation posts or places where measurements are being made, and evolution data would need to be predicted and compared with future measurements. Unlike traditional problems where a grid or mesh is given and the initial condition is assumed given as a function, here we only have the knowledge of the initial values on the arbitrarily distributed point cloud. Hence, it is difficult to apply the classical well developed grid- or mesh-based computational methods to this problem directly, such as the finite difference (FD) methods, the finite volume (FV) methods, and the finite element (FE) methods. Meshless methods [1, 13] are alternatives to traditional mesh-based methods. They provide numerical solutions

Received by the editors December 26, 2016.
2000 *Mathematics Subject Classification.* 65M12, 65M15, 65M60.

in terms of nodes without using any mesh to connect them or using a background mesh only minimally. However, to our best knowledge, there are few papers devoted to meshless methods for solving time-dependent hyperbolic conservation laws [18], and conservation and stability appear to be particularly difficult for meshless methods for such PDEs.

Recently, we designed a high order stable method for this problem in [9]. In order to utilize traditional mesh-based methods which have many important good properties, we provided a way to generate a suitable mesh based on the given point cloud. Each cell in the mesh is a polygon, and contains a minimum number of points in the original point cloud so that a polynomial of a pre-defined degree can be constructed to represent the initial condition to high order accuracy. Once the polygonal mesh is constructed, we march the piecewise polynomial numerical solution in time by choosing the classical discontinuous Galerkin (DG) methods. Due to the good properties of the DG method, the new constructed method is conservative, stable and high order accurate, both for linear and nonlinear equations.

The main difficulty in solving the conservation laws (1) is that solutions may contain discontinuities even if the initial conditions are smooth. As we can see in the numerical examples in [9], when dealing with solutions containing strong shocks, our current scheme on the polygonal mesh will generate spurious numerical oscillations, just as DG schemes without limiters on regular triangular or rectangular meshes will do. These spurious oscillations may lead to nonlinear instability and eventual blow-ups of the codes. Therefore, we need to apply nonlinear limiters to control these oscillations for our polygonal mesh.

To achieve the full potential of high order accuracy and efficiency of our method, we would like to find a robust high order limiting procedure to simultaneously maintain uniform high order accuracy in smooth regions and control spurious numerical oscillations near discontinuities. There are many successful works based on the WENO methodology [10, 11] for DG methods on two-dimensional unstructured triangular meshes, which would serve such a purpose. Zhu et al. [26] designed limiters using the usual WENO reconstruction. They use the cell averages in an adaptive stencil to reconstruct the values of the solutions at certain points in the target cell. Note that the DG method is compact, that is, it uses the information only from the target cell and its immediate neighboring cells. However, the reconstruction stencil in [26] contains not only the immediate neighboring cells of the target cell but also the neighbors' neighbors. To reduce the width of the reconstruction stencil, [14] adopted a Hermite type WENO procedure, which uses not only the cell averages but also the first derivative or first order moment information in the stencil. However the information of neighbors' neighbors is still needed for higher order methods. Also, it is complicated to perform the usual WENO procedure or the Hermite type WENO procedure on unstructured meshes, with the possibility of negative linear weights, as we would need to use extra special treatments to handle them [19]. Recently, a new and simple WENO limiter [27] was designed. This WENO limiter attempts to reconstruct the entire polynomial on the target cell, instead of reconstructing point values or moments in the classical WENO reconstructions. In fact, the entire reconstruction polynomial is just a convex combination of polynomials on the target cell and its immediate neighboring cells (with suitable adjustments for conservation). Hence, it will not harm the compactness of the DG method. Also, the linear weights are always positive.

All the above mentioned WENO limiters are designed for DG methods on triangular meshes. In our method [9] to solve conservation laws on random points, the