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An Energy Conserving Local Discontinuous Galerkin Method for a Nonlinear Variational Wave Equation

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Abstract. We design and numerically validate a local discontinuous Galerkin (LDG) method to compute solutions to the initial value problem for a nonlinear variational wave equation originally proposed to model liquid crystals. For the semi-discrete LDG formulation with a class of alternating numerical fluxes, the energy conserving property is verified. A dissipative scheme is also introduced by locally imposing some numerical "damping" in the scheme so to suppress some numerical oscillations near solution singularities. Extensive numerical experiments are presented to validate and illustrate the effectiveness of the numerical methods. Optimal convergence in L^2 is numerically obtained when using alternating numerical fluxes. When using the central numerical flux, only sub-optimal convergence is observed for polynomials of odd degree. Numerical simulations with long time integration indicate that the energy conserving property is crucial for accurately capturing the underlying wave shapes.

AMS subject classifications: 65M60, 65M12, 35Q35

Key words: Discontinuous Galerkin method, variational wave equation, energy conservation.

1 Introduction

Many applications involve the solution of wave equations. In this paper, we consider a variational nonlinear wave equation that models the propagation of orientation waves in the director field in nematic liquid crystals. Let the director filed $n(x,t)$ be the orientation of the molecules at each location x and time t, in planar deformations of nematic liquid crystals involving only one-dimensional space variable, the director field is given by

$$n(x,t)=\cos u(x,t)e_x+\sin u(x,t)e_y,$$

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where e_x and e_y are the coordinator vectors in the x and y directions, respectively. In such a setting, the dynamics of the liquid crystal is described by some unknown function $u(x,t)$, which represents the angle of the director field relative to the x-direction, and the variational principle [28] reduces to the following nonlinear wave equation

$$\partial_t^2 u - c(u)\partial_x(c(u)\partial_x u) = 0, \tag{1.1}$$

and the wave speed $c(u)$ is given by

$$c^2(u) = \alpha\cos^2 u + \beta\sin^2 u$$

for some positive constant α,β. In general, we assume that $c(u)$ is a smooth, uniformly positive function. One of the most important properties of the wave equation is the conservation of energy. Indeed, the analysis in [5,6,24] shows that conservative solutions are unique, globally defined, and yield a flow on the space of couples $(u,u_t)\in H^1(\mathbb{R})\times L^2(\mathbb{R})$. For each conservative solution, the total energy

$$E(t) = \frac{1}{2}\int (\partial_t u(x,t)^2 + (c(u(x,t))\partial_x u(x,t))^2)dx$$

remains constant in time. Aim of this work is to compute such a conservative solution to (1.1) with an arbitrary high order of accuracy.

Note that smooth solutions may well develop singularities in finite time [17]. It was observed in [6, 24] that conservative solutions can occasionally be measure-valued. On the other hand, for dissipative solutions, studied in [7, 17, 33, 34], the continuous dependence for general initial data in $H^1\times L^2$ remains an open question. Nevertheless, when singularity occurs in solutions, the numerical approximation to u_x may become oscillatory. We therefore also propose a dissipative scheme by locally imposing some numerical "damping" in the conservative scheme, in such a way that it not only indicates where the singularity is located, but it also provides a measure for the artificial damping that smoothens singularities. Relatively little dissipation is added in the smooth regions away from the singularity to ensure an accurate computation of solution structures away from singularities.

Existing numerical results for wave propagation reveal that energy conserving numerical methods, which conserve the discrete approximation of energy, are favorable because they are able to maintain the phase and shape of the waves accurately. Numerical methods without this property may result in substantial phase and shape errors after long time integration.

A vast amount of literature can be found on the numerical approximation of wave equations, including finite difference, finite element, finite volume, spectral methods and integral equation based methods. In this paper, we will confine our attention to the discontinuous Galerkin (DG) method, which is a class of finite element methods using completely discontinuous piecewise-polynomial space for the numerical solution and the test functions in the spatial variables. Various DG methods have been designed for first order