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# EXPONENTIAL FOURIER COLLOCATION METHODS FOR SOLVING FIRST-ORDER DIFFERENTIAL EQUATIONS*

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**Abstract**

In this paper, a novel class of exponential Fourier collocation methods (EFCMs) is presented for solving systems of first-order ordinary differential equations. These so-called exponential Fourier collocation methods are based on the variation-of-constants formula, incorporating a local Fourier expansion of the underlying problem with collocation methods. We discuss in detail the connections of EFCMs with trigonometric Fourier collocation methods (TFCMs), the well-known Hamiltonian Boundary Value Methods (HBVMs), Gauss methods and Radau IIA methods. It turns out that the novel EFCMs are an essential extension of these existing methods. We also analyse the accuracy in preserving the quadratic invariants and the Hamiltonian energy when the underlying system is a Hamiltonian system. Other properties of EFCMs including the order of approximations and the convergence of fixed-point iterations are investigated as well. The analysis given in this paper proves further that EFCMs can achieve arbitrarily high order in a routine manner which allows us to construct higher-order methods for solving systems of first-order ordinary differential equations conveniently. We also derive a practical fourth-order EFCM denoted by EFCM(2,2) as an illustrative example. The numerical experiments using EFCM(2,2) are implemented in comparison with an existing fourth-order HBVM, an energy-preserving collocation method and a fourth-order exponential integrator in the literature. The numerical results demonstrate the remarkable efficiency and robustness of the novel EFCM(2,2).

*Mathematics subject classification:* 65L05, 65L20, 65M20, 65M70.

*Key words:* First-order differential equations, Exponential Fourier collocation methods, Variation-of-constants formula, Structure-preserving exponential integrators, Collocation methods.

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## 1. Introduction

This paper is devoted to analysing and designing novel and efficient numerical integrators for solving the following first-order initial value problems

$$u'(t) + Au(t) = g(t, u(t)), \qquad u(0) = u_0, \qquad t \in [0, t_{\text{end}}], \tag{1.1}$$

where $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is an analytic function, $A$ is assumed to be a linear operator on a Banach space $X$ with a norm $\|\cdot\|$, and $(-A)$ is the infinitesimal generator of a strongly continuous semigroup $e^{-tA}$ on $X$ (see, e.g. [27]). This assumption of $A$ means that there exist two constants $C$ and $\omega$ satisfying

$$\left\|e^{-tA}\right\|_{X\leftarrow X} \leq Ce^{\omega t}, \qquad t \geq 0. \tag{1.2}$$

An analysis about this result can be found in [27]. It is noted that if $X$ is chosen as $X = \mathbb{R}^d$ or $X = \mathbb{C}^d$, then the linear operator $A$ can be expressed by a $d \times d$ matrix. Accordingly in this case, $e^{-tA}$ is exactly the matrix exponential function. It also can be observed that the condition (1.2) holds with $\omega = 0$ provided the field of values of $A$ is contained in the right complex half-plane. In the special and important case where $A$ is skew-Hermitian or Hermitian positive semidefinite, we have $C = 1$ and $\omega = 0$ in the Euclidean norm, independently of the dimension $d$. If $A$ originates from a spatial discretisation of a partial differential equation, then the assumption of $A$ leads to temporal convergence results that are independent of the spatial mesh.

It is known that the exact solution of (1.1) can be represented by the variation-of-constants formula

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}g(\tau, u(\tau))d\tau. \tag{1.3}$$

For oscillatory problems, the exponential subsumes the full information on linear oscillations. This class of problems (1.1) frequently rises in a wide variety of applications including engineering, mechanics, quantum physics, circuit simulations, flexible body dynamics and other applied sciences (see, e.g. [10,16,24,27,39,41,44,47]). Parabolic partial differential equations with their spatial discretisations and highly oscillatory problems are two typical examples of the system (1.1) (see, e.g. [30–34,42]). Linearizing stiff systems $u'(t) = F(t, u(t))$ also yields examples of the form (1.1) (see, e.g. [15,25,28]).

Based on the variation-of-constants formula (1.3), the numerical scheme for (1.1) is usually constructed by incorporating the exact propagator of (1.1) in an appropriate way. For example, interpolating the nonlinearity at the known value $g(0, u_0)$ yields the exponential Euler approximation for (1.3). Approximating the functions arising by rational approximations leads to implicit or semi-implicit Runge–Kutta methods, Rosenbrock methods or W-schemes. Recently, the construction, analysis, implementation and application of exponential integrators have been studied by many researchers, and we refer the reader to [3,11–13,16,37,45], for example. Exponential integrators make explicit use of the quantity $Au$ of (1.1), and a systematic survey of exponential integrators is referred to [27].

Based on Lagrange interpolation polynomials, exponential Runge-Kutta methods of collocation type are constructed and their convergence properties are analysed in [26]. In [40], the authors developed and researched a novel type of trigonometric Fourier collocation methods (TFCMs) for second-order oscillatory differential equations $q''(t) + Mq(t) = f(q(t))$ with a principal frequency matrix $M \in \mathbb{R}^{d\times d}$. These new trigonometric Fourier collocation methods