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FINITE ELEMENT METHODS FOR THE NAVIER-STOKES EQUATIONS BY $H(\text{div})$ ELEMENTS*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

We derived and analyzed a new numerical scheme for the Navier-Stokes equations by using $H(\text{div})$ conforming finite elements. A great deal of effort was given to an establishment of some Sobolev-type inequalities for piecewise smooth functions. In particular, the newly derived Sobolev inequalities were employed to provide a mathematical theory for the $H(\text{div})$ finite element scheme. For example, it was proved that the new finite element scheme has solutions which admit a certain boundedness in terms of the input data. A solution uniqueness was also possible when the input data satisfies a certain smallness condition. Optimal-order error estimates for the corresponding finite element solutions were established in various Sobolev norms. The finite element solutions from the new scheme feature a full satisfaction of the continuity equation which is highly demanded in scientific computing.

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1. Introduction

We are concerned with numerical solutions of the Navier-Stokes equations: find a pair of unknown functions $(\mathbf{u}; p)$ satisfying

$$-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where ν denotes the fluid viscosity; Δ, ∇, and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators, respectively; $\Omega \subset \mathbb{R}^n$ is the region occupied by the fluid; $\mathbf{f} = \mathbf{f}(\mathbf{x}) \in [L^2(\Omega)]^n$ is the unit external volumetric force acting on the fluid at $\mathbf{x} \in \Omega$.

The commonly used finite element methods for the Navier-Stokes problem (1.1)-(1.3) are based on a variational equation which is obtained by testing the momentum equation (1.1) by functions in $[H_0^1(\Omega)]^n$ and the continuity equation (1.2) by functions in $L^2(\Omega)$ (see Section 2 for their definition). The corresponding finite element method requires a pair of finite element spaces which are conforming in $H^1 \times L^2$ and satisfy the *inf-sup* condition of Babuška [3] and

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Brezzi [4]. These constraints result in finite element approximations, denoted by $(\mathbf{u}_h; p_h)$, which hardly satisfy the continuity equation

$$\nabla \cdot \mathbf{u}_h(\mathbf{x}) = 0, \quad \forall\, \mathbf{x} \in \Omega. \tag{1.4}$$

Readers are referred to [19] and [9] for more details regarding the approximation methods and their properties.

The recent development in discontinuous Gelerkin methods [2, 5–7, 11, 12, 14] provides new means in solving the incompressible problems numerically. However, the corresponding finite element solutions are usually totally discontinuous and fail to satisfy the continuity equation (1.4) immediately [13, 23, 26, 30].

Eq. (1.4) requires that the numerical solution $\mathbf{u}_h$ be a member of the Sobolev space $H(\text{div}; \Omega)$. Therefore, the discontinuous Galerkin methods [13,23,26,30] may not be appropriate when (1.4) needs to be satisfied. On the other hand, the $H^1 \times L^2$ conforming finite element methods require the total continuity of $\mathbf{u}_h$, which is beyond what is required for a satisfaction of (1.4). Therefore, it appears that the $H(\text{div})$ elements of Raviart-Thomas type [27] might be appropriate for approximating the solution of the Navier-Stokes equations.

In [29], a finite element scheme for the Stokes equations was derived and analyzed by using existing $H(\text{div})$ finite elements of the Raviart-Thomas type. The numerical solutions of the finite element schemes developed in [29] satisfy the incompressibility constraint (1.2) exactly. The goal of this paper is to continue our investigation in $H(\text{div})$ finite element methods by extending the results of [29] to the Navier-Stokes equations. There are two main difficulties in this extension. The first one lies on a treatment of the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ in designing a numerical discretization scheme for (1.1)-(1.3). An up-winding approach shall be used to tackle this difficulty, yielding a numerical scheme that should be stable for small viscosities. The second difficult is associated with a mathematical analysis for the numerical scheme; namely, one has to deal with the difficulties caused by discontinuity of the finite elements and the corresponding integral forms over the element boundaries. Some Sobolev-type inequalities are established to address this challenge.

This paper is organized as follows. In Section 2, we introduce some preliminaries and notations for Sobolev spaces. A variational formula is presented in Section 3 for the Navier-Stokes equations. In Section 4, we present a $H(\text{div})$ finite element method for the Navier-Stokes equations, based on the variational formula developed in Section 3. In Section 5, we derive some Sobolev-type inequalities for piecewise smooth functions. Section 6 is devoted to a mathematical study of the finite element scheme. Here it was proved that the new finite element scheme has solutions and the solutions are unique when the input data is sufficiently small. In Section 7, we establish some optimal-order error estimates for the finite element approximations in a discrete H^1-norm for the velocity approximation and L^2-norms for the pressure.

2. Preliminaries and Notations

Let D be any domain in $\mathbb{R}^n, n = 2, 3$. For simplicity, we take the case $n = 2$ as a protocol in the presentation and analysis. Extension to problems in three space variables is possible for all the results to be presented in this manuscript.

We use standard definitions for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot, \cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and seminorms $|\cdot|_{s,D}$ for $s \geq 0$. For example, for any integer $s \geq 0$,