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A TRUST-REGION ALGORITHM FOR NONLINEAR INEQUALITY CONSTRAINED OPTIMIZATION*[1)]

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Abstract

This paper presents a new trust-region algorithm for n-dimension nonlinear optimization subject to m nonlinear inequality constraints. Equivalent KKT conditions are derived, which is the basis for constructing the new algorithm. Global convergence of the algorithm to a first-order KKT point is established under mild conditions on the trial steps, local quadratic convergence theorem is proved for nondegenerate minimizer point. Numerical experiment is presented to show the effectiveness of our approach.

Key words: Inequality constrained optimization; Trust-region method; Global convergence; Local quadratic convergence.

1. Introduction

In this paper, we study the following nonlinear inequality constrained optimization problem:

$$\begin{cases} min & f(x) \\ s.t & H(x) \le 0, \end{cases} \tag{1.1}$$

where $H(x) = (h_1(x), h_2(x), \cdots, h_m(x))^T$, f(x) and $h_i(x)$, $i \in I = \{1, 2, \cdots, m\}$, are $R^n \to R$ twice continuously differentiable. We assume $m \le n$ in this paper, which is important for our argument.

Trust-region algorithms are very efficient for solving nonlinear equality constrained problems. (see, [1], [2], [5], [11], for example). However, for nonlinear inequality constrained optimization problem, the results about trust region method are very few, (see [4], [6], [9], [12], [7], for example), and there are still some unsolved problems now. The paper [7] deals with inequality using slack variables and finally only discusses equality constraints and bound constraints. The convergence to so-called φ-stationary point has been proved. The paper [9] presents trust region method for an arbitrary closed set and proves the global convergence theorem. But it is very difficult to solve the subproblems arisen in the algorithm of [9]. Very general problems have been discussed in [12]. The basic idea of [12] is to reduce the smooth constrained optimization problem into a nonsmooth unconstrained problem by using l_∞ exact penalty function and then to solve the nonsmooth problem by trust region method. Global convergence of the method has been proved under the assumption that the penalty parameter is bounded. When the penalty parameter tends to infinity, the method of [12] is still convergent, but the limit is not the KKT point of the original problem. [3] discusses an interior Newton method, [4] gives a trust region approaches for nonlinear optimization only for a special case, that is the optimization problem

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with bounded constraints. [6] extends the method of [3] and [4] to the problem with bound constraints for partial variables and equality constraints.

This paper presents a new trust region method for nonlinear optimization with inequality constraints. We change the problem into an equivalent problem with equality constraints and non-negative constraints by using slack variables. Then we derive a new equivalent KKT conditions which is the basis for constructing our algorithm. The subproblems in the algorithm can be solved by the method proposed in [5] and [6]. We have proved that at least one accumulation point of the new algorithm is a first- order KKT point. The local quadratic convergence for nondegenerate minimizer point has been shown.

The problem proposed by this paper is different from [6], Assume p is the number of bound constraints. [6] requires that $m \leq n$ and $p = n - m$. Hence the problem of [6] can be reduced to an optimization problem with simply bound constraints. In our paper, by introducing slack variables problem (1.1) is changed into problem (2.1), where the number p of bound constraints is m. So our problem is different from the problem in [6].

The paper is organized as follows. In section 2, we derive an equivalent first-order KKT condition; In section 3, we discuss the Newton's method of the KKT equations; We present a method to compute trial step in section 4; In section 5, the new trust-region algorithm is formulated; The global convergence theorem of the algorithm is given in section 6; Section 7 makes local analysis for the algorithm; The last section is numerical test.

In this paper, the vector and matrix norms used are l_2 norm, subscripted indices k represents the evaluation of a function at a particular point. For example, f_k represents $f(x_k)$ and so on.

2. Optimality conditions

By introducing slack variables $s \in R^m$, (1.1) is transformed equivalently to the following problem for the variables $x \in R^n$ and $s \in R^m$:

$$\begin{cases} min & f(x) \\ s.t & H(x) + s = 0, \quad s \geq 0. \end{cases} \tag{2.1}$$

Denote $u = (x^T, s^T)^T \in R^{m+n}, C(u) = H(x) + s \in R^m$,

$$l(x, s, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i (h_i(x) + s_i),$$

$$A(x) = (\nabla h_1(x), \nabla h_2(x), \cdots, \nabla h_m(x)) \in R^{n \times m},$$

$J(u) = (A^T(x), I_m) \in R^{m \times (n+m)}$, which is Jacobian of $C(u)$. $I_m \in R^{m \times m}$is unit matrix.

A point $u^* = ((x^*)^T, (s^*)^T)^T$ satisfies the first -order KKT conditions of problems (2.1) if there exist $\lambda^* \in R^m$ such that:

$$\begin{cases} H(x^*) + s^* = 0, \quad s^* \geq 0, \\ \nabla f(x^*) + A(x^*)\lambda^* = 0, \\ s_i^* > 0 \quad \Longrightarrow \lambda_i^* = 0; \quad s_i^* = 0 \quad \Longrightarrow \lambda_i^* \geq 0 \quad i \in I. \end{cases} \tag{2.2}$$

We assume $m \leq n, rankA(x) = m$ in this paper. So the constrained qualification is satisfied. From QR factorization of $A(x)$

$$A(x) = (Y(x), Z(x)) \begin{pmatrix} R(x) \\ 0 \end{pmatrix}, \tag{2.3}$$

(2.2) is equivalent to

$$\begin{array}{l} H(x^*) + s^* = 0, \ s^* \geq 0, \ Z(x^*)^T \nabla f(x^*) = 0, \\ s_i^* > 0 \ \Longrightarrow [-R(x^*)^{-1} Y(x^*)^T \nabla f(x^*)]_i = 0, \\ s_i^* = 0 \ \Longrightarrow [-R(x^*)^{-1} Y(x^*)^T \nabla f(x^*)]_i \geq 0. \end{array} \tag{2.4}$$