Journal of Computational Mathematics, Vol.12, No.1, 1994, 71–77.

A PRECONDITIONER DETERMINED BY A SUBDOMAIN COVERING THE INTERFACE[*1)]

Zhang Sheng
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

A method for construction of a preconditioner for the capacitance matrix on the interface is described. The preconditioner is determined by a subdomain covering the interface, and the condition number of the preconditioned matrix is dependent on the width of the covering subdomain and independent of the discrete mesh size and discontinuity of the coefficients of the differential operator. Some applications of our theory are presented at last.

1. Introduction

Let $\Omega \subset R^2$ be a polygonal region, and

$$Lu = -\sum_{i,j=1}^{2} \frac{\partial u}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + cu.$$

be an elliptic operator defined on it; here, $(a_{i,j})_{i,j=1,2}$ is symmetric positive definite and bounded from above and below on Ω, $c \geq 0$.

$$\begin{cases} a(u,v) = (f,v), \quad v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega) \end{cases} \tag{1.1}$$

is the variational form of the boundary value problem, with the bilinear form

$$a(u,v) = \int_\Omega \Big[\sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + cuv\Big].$$

For convenience we discuss only the homogeneous Dirichlet boundary value problem here. The norm in $H_0^1(\Omega)$ introduced by $a(\cdot,\cdot)$ is equivalent to the original one. $H_0^1(\Omega)$ will be treated as a Hilbert space with inner product $a(\cdot,\cdot)$ in the following.

$(1,1)$ is discretized by the finite element method. Triangulation and linear continuous element will be discussed. The triangulation is supposed to be local and regular. The discrete form of (1.1) is

$$\begin{cases} a(u,v) = (f,v), \quad v \in S_0^h(\Omega), \\ u \in S_0^h(\Omega). \end{cases} \tag{1.2}$$

* Received August 14, 1992.

1) A key project of Chinese Academy of Sciences.

The domain Ω is decomposed into two non-overlapping subdomains Ω_1 and Ω_2 by the interface Γ which coincides with the finite element triangulation. $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2, \bar{\Omega}_1 \cap \bar{\Omega}_2 = \Gamma, \Omega_1 \cap \Omega_2 = \emptyset$.

$\hat{\Omega}$ represents the set of finite element node points in Ω, $\hat{\Omega}_1 = \hat{\Omega} \cap \Omega_1, \hat{\Omega}_2 = \hat{\Omega} \cap \Omega_2, \hat{\Gamma} = \Gamma \cap \hat{\Omega}$. (1.2) may be written in matrix -vector form

$$\begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} \tag{1.3}$$

where x_0, x_1 and x_2 are vectors corresponding to restrictions of the finite element function on $\hat{\Gamma}, \hat{\Omega}_1$ and $\hat{\Omega}_2$. (1.3) can be attributed to a small scale problem on $\hat{\Gamma}$ by Gauss elimlnation

$$Cx_0 = d \tag{1.4}$$

where

$$C = A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20},$$

$$d = b_0 - A_{01}A_{11}^{-1}b_1 - A_{02}A_{22}^{-1}b_2,$$

and C is the capacitance matrix or Schur complement.

When (1.4) is solved, (1.3) will become two isolated Dirichlet boundary value problems. The iterative method is often used to solve (1.4). Since Cond $(C) = O(h^{-1})$, a proper preconditioner is necessary. There are many preconditioners constructed in recent years ([1–4] and the probing technique in [4]), but those preconditioners are proved or verified numerically to be spectrally equivalent to the capacitance matrix, but it should be noted that the condition number of the preconditioned matrix by these preconditioners will depend on the shape, size of Ω, Ω_1 and Ω_2 and the coefficients of the differential operator.

We will construct a new preconditioner in this paper, which is determined by a subdomain covering the interface. The condition number of the preconditioned matrix is independent of the subdomains and the interface $(\Omega_1, \Omega_2, \Gamma)$ and discontinuity of the coefficients of the differential operator, which is determined by the width of the covering subdomain.

2. A Preconditioner Determined by a Subdomain Covering the Interface

Ω_0 is a subdomain of Ω covering Γ, the boundary of which coincides with the finite element mesh line. Ω_0 has a uniform overlap with Ω_1 and Ω_2; the width of Ω_0 is of order $O(\delta)$. $\Omega_{01} = \Omega_0 \cap \Omega_1$, $\Omega_{02} = \Omega_0 \cap \Omega_2$.

$\{\phi_i, i \in \hat{\Omega}\}$ is the set of the usual finite element basis functions. The element of the stiffness matrix is

$$a_{ij} = a(\phi_i, \phi_j), \quad i, j \in \hat{\Omega};$$